

OGAP Additive Framework

Depending upon the strength of additive reasoning, students may move up and down between additive, transitional, counting, and non-additive reasoning and strategies as they interact with new topics or new concepts.

Problem Contexts

Counting

Quantities
Patterns

Additive Situations

Add to
Take from
Put together/Take apart
Compare

Concepts/Properties

Equations
Properties and Relationships
Magnitude

Base 10/Place Value

Concepts
Strategies

Problem Structures

Types of Items

Contextual
Non-contextual

Complexity of Addends

Single digit
Multiple digit
Multiples of 10, 100, 1000

Representations

Number Paths and Number Line
Other length-based models
Base 10
Ten Frames
Dot images

Properties and Relationships

Additive inverse
Commutative property
Associative property
Identity
Relationships between models, equations and contexts
Compensation
Constant difference

Addends

Two addends
Three or more addends
Compositions of 10

Meanings for Subtraction

Difference
Removal
Distance
Missing addend

Number of steps

Single step
Multi-step

	Result Unknown	Change Unknown	Start Unknown
Add To	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? $2 + 3 = ?$	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? $2 + ? = 5$	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? $? + 3 = 5$
Take From	Five apples were on the table. I ate two apples. How many apples are on the table now? $5 - 2 = ?$	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? $5 - ? = 3$	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? $? - 2 = 3$
	Total Unknown	Addend Unknown	Both Addends Unknown
Put Together/ Take Apart	Three red apples and two green apples are on the table. How many apples are on the table? $3 + 2 = ?$	Five apples are on the table. Three are red and the rest are green. How many apples are green? $3 + ? = 5$ $5 - 3 = ?$	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? $5 = 0 + 5$ $5 = 5 + 0$ $5 = 1 + 4$ $5 = 4 + 1$ $5 = 2 + 3$ $5 = 3 + 2$
	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare	("How many more?" version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? ("How many fewer?" version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? $2 + ? = 5$ $5 - 2 = ?$	(Version with "more"): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with "fewer"): Lucy has three fewer apples than Julie. Lucy has two apples. How many apples does Julie have? $2 + 3 = ?$ $3 + 2 = ?$	(Version with "more"): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with "fewer"): Lucy has three fewer apples than Julie. Julie has five apples. How many apples does Lucy have? $5 - 3 = ?$ $? + 3 = 5$

Reproduced from the CCSSM (2010) Table 1 Common addition and subtraction situations.

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As students learn new concepts or interact with new problem situations and problems structures, they may move up and down along this progression.

Additive

Additive Strategies

Traditional US algorithm

$$\begin{array}{r} 1 \\ 18 \\ +27 \\ \hline 45 \end{array}$$

Transparent algorithms

Decomposes by place value and adds

$$18 + 27 = ?$$

$$18 = 10 + 8$$

$$18 + 20 = 38$$

$$27 = 20 + 7$$

$$38 + 7 = 45$$

$$30 + 15 = 45$$

Partial Sums

$$\begin{array}{r} 18 \\ +27 \\ \hline 30 \\ +15 \\ \hline 45 \end{array}$$

Uses properties

Associative Property
 $8 + 7 + 2 = (8 + 2) + 7 = 10 + 7$

Commutative Property
 $6 + 7 = 7 + 6$

Flexible compensation

$$\begin{array}{r} 18 + 27 \\ 20 + 25 \\ \hline 45 \end{array}$$

Fact Fluency

Fact Recall
 $6 + 7 = 13$

Derived Facts
 $6 + 6 = 12$
 $6 + 7 = 13$

Using 10s

$6 + 7$
 $6 + 4 = 10$ and $10 + 3 = 13$

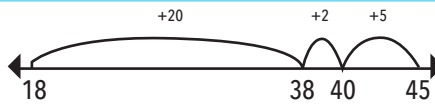
Transitional (Tens)

Transitional Strategies

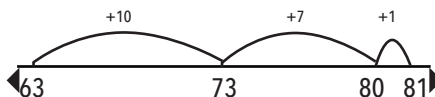
Efficient Use of a Model

$$\begin{array}{r} 18 + 27 \\ \hline 45 \end{array}$$

Decomposed into $10 + 8$ and $20 + 7$, then combined into $30 + 15 = 45$.



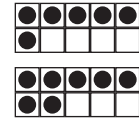
Jumps by multiples of 10 on a number line



Jumps by a 10 and efficient groups of ones

Inefficient decomposition to derive facts

$6 + 7$
 "I know $5 + 5$ is 10 and 3 more is 13"

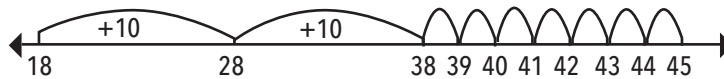


Unitizes on a model

Early Transitional Strategies

Adding inefficiently with or without a model

Adding on by tens
 $18 + 10 = 28$
 $28 + 10 = 38$
 $38 + 2 = 40$
 $40 + 5 = 45$



Makes jumps of 10 on a number line

Combines or counts by 10s using base 10 representations



Counting (Ones)

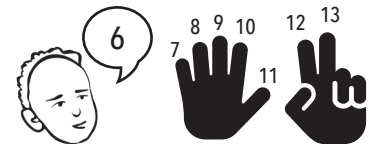
Counting Strategies

Mental counting strategies:

- Count on from first
- Count on from larger



Counting on with physical or visual model: $6 + 7$



"7, 8, 9, 10, 11, 12, 13"

Early Counting Strategies

Direct modeling and counting from 1 with concrete model (count 3 times)



"1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13"

Non-Additive Strategies

- Uses incorrect operation
- Models problem situation incorrectly
- Guesses
- Not enough information
- Uses procedures incorrectly

Underlying Issues/Errors

- Does not consider reasonableness of solution
- Error in counting, calculation, place value, property, equation, or model
- Units inconsistent or missing

Subitizing, Unitizing, Commutativity, Number Composition, Place value understanding



Additive Reasoning Progression – Subtraction

As students learn new concepts or interact with new problem situations and problems structures, they may move up and down along this progression.

Subitizing, Unitizing, Commutativity, Number Composition, Base 10 understanding

Additive

Additive Strategies

Traditional US algorithm

$$\begin{array}{r} 6 \ 1 \\ 73 \\ -38 \\ \hline 35 \end{array}$$

Transparent algorithms

Decomposes by place value and subtracts

$$\begin{array}{l} 73 - 38 = ? \\ 73 = 70 + 3 \qquad 73 - 30 = 43 \\ 38 = 30 + 8 \qquad 43 - 3 = 40 \\ 40 - 5 = 35 \qquad 40 - 5 = 35 \end{array}$$

Adding Up Efficiently

$$\begin{array}{l} 38 + 2 = 40 \\ 40 + 33 = 73 \\ 2 + 33 = 35 \end{array}$$

Constant Difference:

$$\begin{array}{l} 73 - 38 \\ +2 \ +2 \\ 75 - 40 = 35 \end{array}$$

Partial Difference:

$$\begin{array}{l} 73 \\ - 38 \\ \hline 40 - 5 = 35 \end{array}$$

Fact Fluency

Fact Recall

Derived Facts

Inverse relationship between addition and subtraction

"7 - 2 is 5 because 5 + 2 is 7"

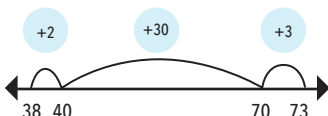
Using 10s: 13 - 8

"13 - 3 = 10 and 10 - 5 = 5"

Transitional (Tens)

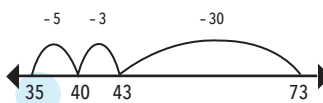
Transitional Strategies

Efficient Use of a Model



$$73 - 38 = 35$$

Distance on number line



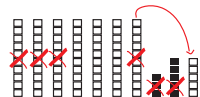
$$73 - 38 = 35$$

Jump back on number line

Early Transitional Strategies

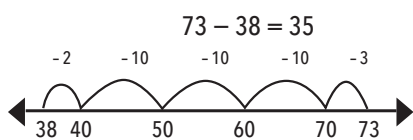
Subtracting inefficiently with or without a model

Using Base 10 models

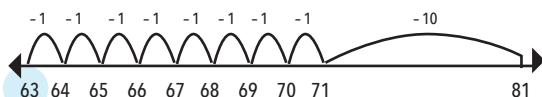


$$73 - 38 = 35$$

Jumps by tens on number line

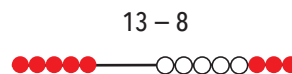


$$81 - 18 = 63$$



Adding up by tens

$$\begin{array}{l} 73 - 38 = ? \\ 38 + 2 = 40 \\ 40 + 10 = 50 \\ 50 + 10 = 60 \\ 60 + 10 = 70 \\ 70 + 3 = 73 \\ 2 + 10 + 10 + 10 + 3 = 35 \end{array}$$



"13 - 3 is 10 and then 5 less is 5"

Unitizes on a model

Counting (Ones)

Counting Strategies

Counting back (7 - 5)

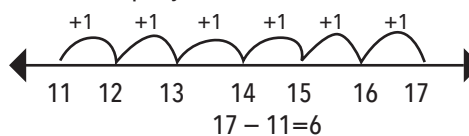
"7...6,5,4,3,2"



Counting back (7 - 5)

"7...6,5...that's 2"

Jumps by ones on number line



Early Counting Strategies

Direct modeling—separating from, separating to, or matching (counts 3 times)



Non-Additive Strategies

- Uses incorrect operation
- Models problem situation incorrectly
- Guesses
- Not enough information
- Uses procedures incorrectly

Underlying Issues/Errors

- Does not consider reasonableness of solution
- Error in counting, calculation, place value, property, equation, or model

As students learn new concepts or interact with new problem situations and problems structures, they may move up and down along this progression.

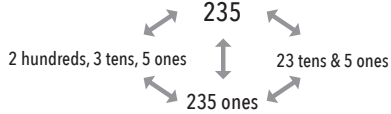
Base Ten

Flexible Use of Base Ten

Multiplicative Place Value

$$45 = (4 \times 10) + 5 \quad 12 \text{ tens and 3 ones is } 123$$

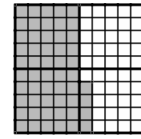
Uses integrated Base Ten understanding



Applies Base Ten understanding to compare and solve problems

"63 > 59 because 6 tens is more than 5 tens"

Conceptual subitizing with place value



5 tens + 4

Number Composition

Number Composition (Unitizing in tens)

Composition and decomposition

$$56 = 50 + 6$$

$$200 + 30 + 5 = 235$$

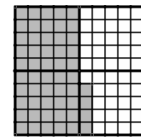
Unitizing by 10's and 1's with understanding
10, 20, 30, 40, 41, 42, 43

Using number relationships to compare and solve problems

"6 is 2 more than 4"

Applies counting by 2's and 5's to solve problems

Conceptual subitizing with 5's and 10's



10, 20, 30, 40, 50, 54

Flexible Counting

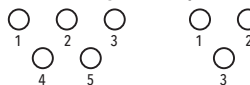
Flexible Counting (Unitizing in small groups)

Counting unseen objects



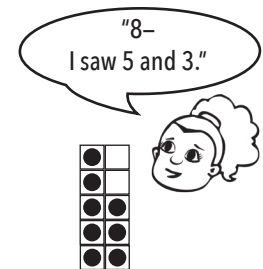
5, 6, 7, 8

Counting to compare



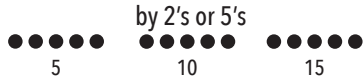
"5 is more than 3"

Conceptual subitizing

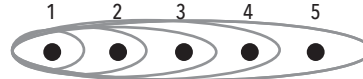


Counting on or back from a given number

Conceptual understanding of counting by 2's or 5's



One more/less than (without counting from 1)



Early Counting

Early Counting (Sees quantities as collections of ones)

Cardinality (Meaningful counting)

"How many?"

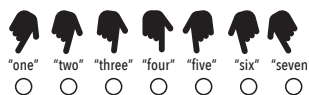


Matching to compare



Counts by ones to determine quantity

Counting with one-to-one correspondence

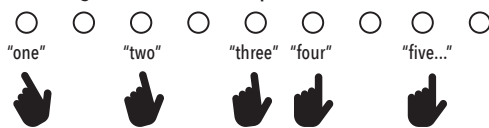


Representing an amount by ones

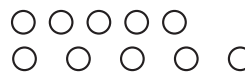
Precounting

Precounting

Counting with some correspondence



Comparing quantity by size



"The second row is bigger"

Perceptual subitizing up to 4



Rote counting

Underlying Issues/Errors

- Doesn't consider reasonableness of solution
- Sequence error
- Errors in units
- Numeral reversal
- Digit reversal
- Counting Error

Subitizing, Mental Representations, Conservation, Base Ten Understanding