



Formative Assessment Tools and Routines for Additive Reasoning

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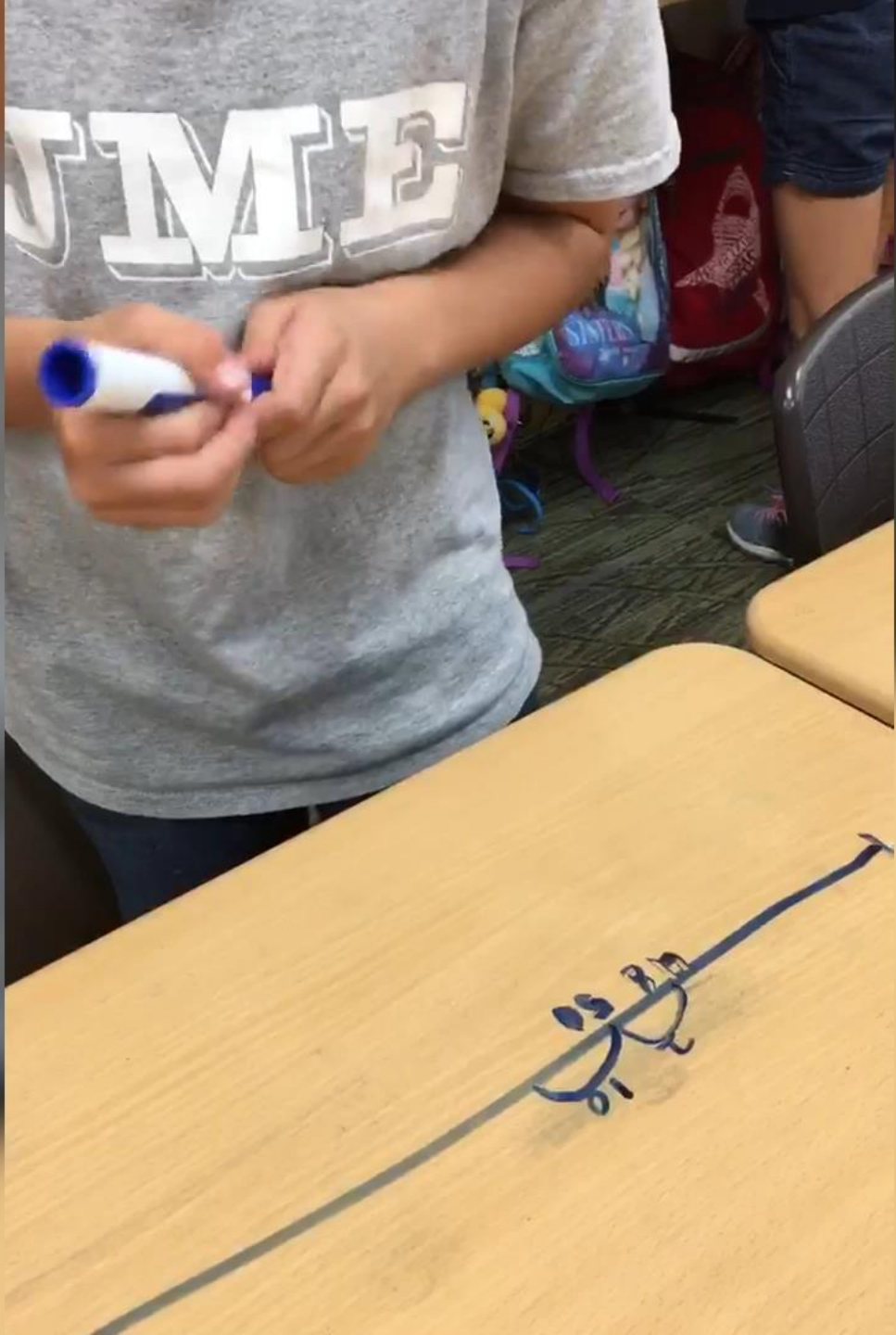
Formative Assessment Tools and Routines for Additive Reasoning



Background
on the
Ongoing
Assessment
Project
(OGAP)

Using the
Additive
Reasoning
progression to
analyze and
respond to
student work

Developing
fluency with
number lines





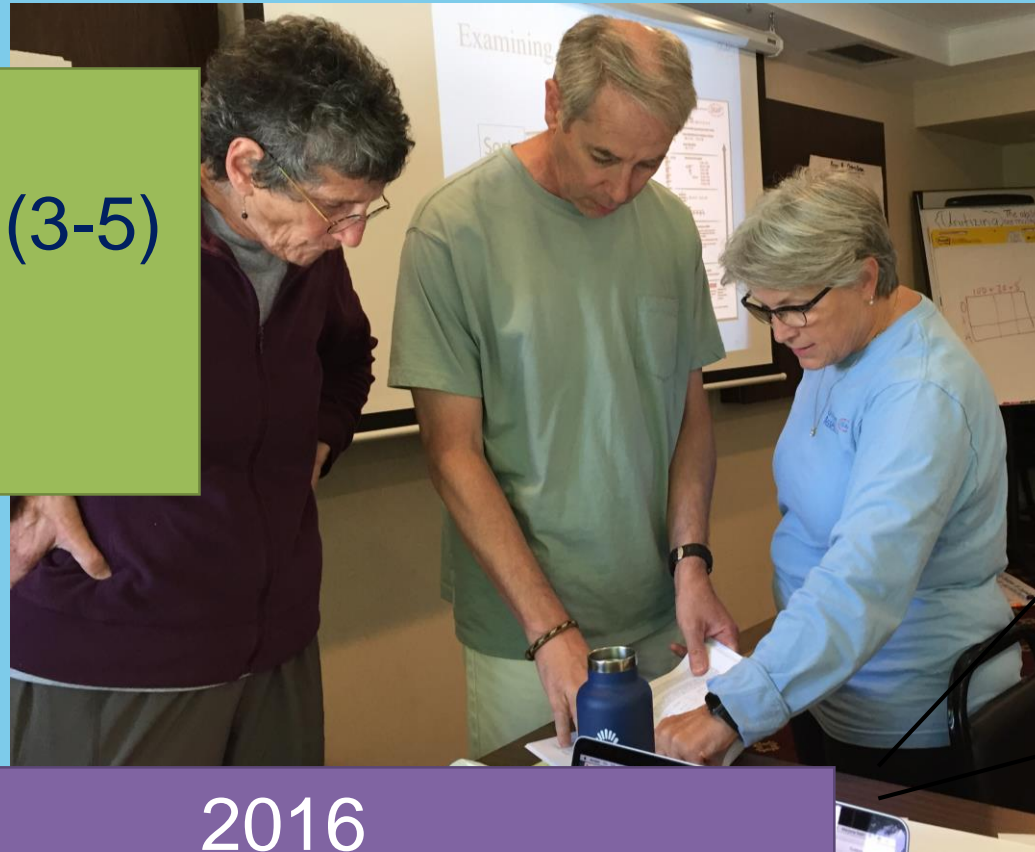
OGAP is...

- An intentional and systematic mathematics formative assessment system
- Based on math education research
- Focused on:
 - how students develop understanding of concepts and procedural fluency,
 - errors or pre-conceptions or misconceptions that may interfere with learning new concepts or solving related problems.



2003-2015

- ✓ Multiplicative Reasoning (3-5)
- ✓ Fractions (3-5)
- ✓ Proportionality (6-8)



2016

Additive Reasoning (K-2)

Alabama

Vermont

Pennsylvania

Maryland

South Carolina



Formative Assessment System

- **Extensive Professional Development**

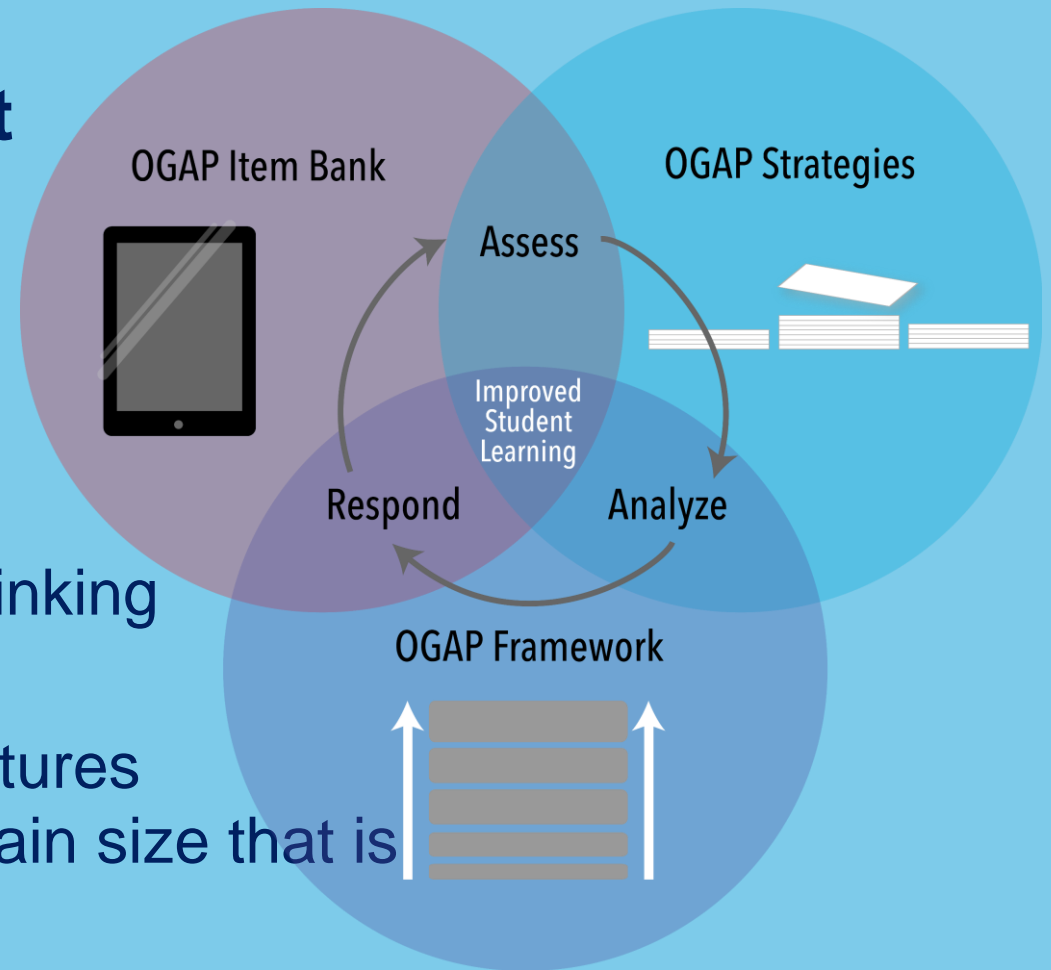
- Content
- Research on student learning
- Formative assessment routines

- **Formative Assessment Items**

- Aligned to CCSSM
- Engineered to elicit evidence of student thinking

- **Frameworks**

- Information on problem contexts and structures
- Learning Progressions/Trajectories at a grain size that is usable by teachers



Coral and Sean are counting crayons.
 Coral counted 15 blue crayons.
 Sean counted 9 red crayons.
 How many crayons did they count together?

Shantelle

$$15 + 9 = 24$$

$$10 + 9 = 19$$

$$19 + 5 = 24$$

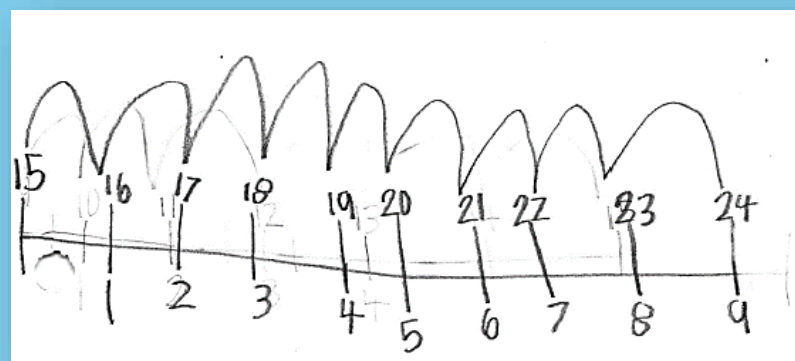
Rachel

1 2 3 4 5
6 7 8 9

10 11 12 13 14
15 16 17 18 19
20 21 22 23 24

24 $9 + 15 = 24$

Declan



Additive

Transitional

Counting

Non-Additive

As students learn new concepts or interact with new problem situations and problems structures, they may move up and down along this progression.

Additive

Transitional (Tens)

Counting (Ones)

Subitizing, Unitizing, Commutativity, Number Composition, Place value understanding

Additive Strategies

<p>Traditional US algorithm</p> $\begin{array}{r} 18 \\ +27 \\ \hline 45 \end{array}$ <p>Transparent algorithms Decomposes by place value and adds</p> $\begin{array}{l} 18 + 27 = ? \\ 18 = 10 + 8 \\ 27 = 20 + 7 \\ 30 + 15 = 45 \end{array}$	<p>Partial Sums</p> $\begin{array}{r} 18 \\ +27 \\ \hline 30 \\ +15 \\ \hline 45 \end{array}$	<p>Uses properties</p> <p>Associative Property $8 + 7 + 2 = (8 + 2) + 7 = 10 + 7$</p> <p>Commutative Property $6 + 7 = 7 + 6$</p>	<p>Fact Fluency</p> <p>Fact Recall $6 + 7 = 13$</p> <p>Derived Facts $6 + 6 = 12$ $6 + 7 = 13$</p> <p>Using 10s $6 + 7$ $6 + 4 = 10$ and $10 + 3 = 13$</p>
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Flexible compensation

$$\begin{array}{r} 18 + 27 \\ 20 + 25 \\ \hline 45 \end{array}$$

Transitional Strategies

Efficient Use of a Model

Jumps by multiples of 10 on a number line

Inefficient decomposition to derive facts

"I know 5 + 5 is 10 and 3 more is 13"

Early Transitional Strategies

Adding inefficiently with or without a model

Adding on by tens

$$\begin{array}{l} 18 + 10 = 28 \\ 28 + 10 = 38 \\ 38 + 2 = 40 \\ 40 + 5 = 45 \end{array}$$

Makes jumps of 10 on a number line

Combines or counts by 10s using base 10 representations

Counting Strategies

Counting on with physical or visual model: 6 + 7

Mental counting strategies:

- Count on from first
- Count on from larger

Early Counting Strategies

Direct modeling and counting from 1 with concrete model (count 3 times)

"1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13"

Non-Additive Strategies

- Uses incorrect operation
- Models problem situation incorrectly
- Guesses
- Not enough information
- Uses procedures incorrectly

Underlying Issues/Errors

- Does not consider reasonableness of solution
- Error in counting, calculation, place value, property, equation, or model
- Units inconsistent or missing

Shantelle

$$15 + 9 = 24$$

$$10 + 9 = 19$$

$$19 + 5 = 24$$

Models that support development of additive reasoning

Declan

Rachel

$$24 \quad 9 + 15 = 24$$

Importance of problem structure and number complexity

As students learn new concepts or interact with new problem situations and problems structures, they may move up and down along this progression.

Additive

Additive Strategies

Traditional US algorithm	Partial Sums	Uses properties	Fact Fluency
$\begin{array}{r} 1 \\ 18 \\ +27 \\ \hline 45 \end{array}$	$\begin{array}{r} 18 \\ +27 \\ 30 \\ +15 \\ \hline 45 \end{array}$	Associative Property $8+7+2=(8+2)+7=10+7$ Commutative Property $6+7=7+6$	Fact Recall $6+7=13$ Derived Facts $6+6=12$ $6+7=13$
Transparent algorithms Decomposes by place value and adds	Flexible compensation		Using 10s
$18 + 27 = ?$ $18 = 10 + 8$ $27 = 20 + 7$ $30 + 15 = 45$	$18 + 27$ $20 + 25$ 45		$6+4=10$ and $10+3=13$

Transitional (Tens)

Transitional Strategies

Efficient Use of a Model

Jumps by multiples of 10 on a number line

Jumps by a 10 and efficient groups of ones

Inefficient decomposition to derive facts

$6+7$
"I know $5+5$ is 10 and 3 more is 13"

Counting (Ones)

Early Transitional Strategies

Adding inefficiently with or without a model

Adding on by tens

$18 + 10 = 28$
 $28 + 10 = 38$
 $38 + 2 = 40$
 $40 + 5 = 45$

Makes jumps of 10 on a number line

Combines or counts by 10s using base 10 representations

Counting Strategies

Counting on with physical or visual model: $6+7$

- Mental counting strategies:
 - Count on from first
 - Count on from larger

Early Counting Strategies

Direct modeling and counting from 1 with concrete model (count 3 times)

Non-Additive Strategies

- Uses incorrect operation
- Models problem situation incorrectly
- Guesses
- Not enough information
- Uses procedures incorrectly

Underlying Issues/Errors

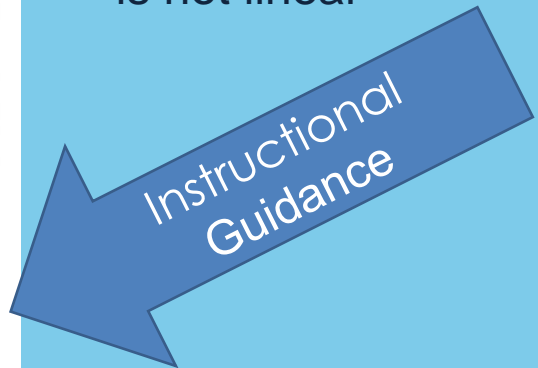
- Does not consider reasonableness of solution
- Error in counting, calculation, place value, property, equation, or model
- Units inconsistent or missing

OGAP Additive Framework

October 2017

Thinking about the strength of additive reasoning, students may move up and down between additive, transitional, counting, and non-additive reasoning and strategies as they interact with new topics or new concepts.

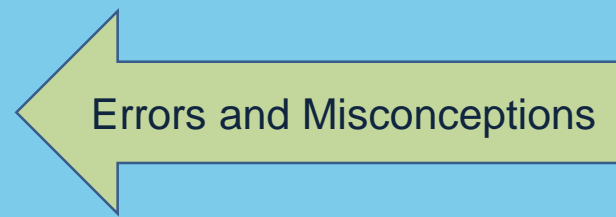
Problem Contexts	Problem Structures	Problem Types
<ul style="list-style-type: none"> Counting Quantities Money Additive Situations Area Measurement Part-Whole Comparison Concepts/Properties Operations and Relationships Measurement Base 10/Place Value Compos Reasoning 	<ul style="list-style-type: none"> Types of Items Contexts Non-contextual Grouping of Addends Grouping Multiplication (10, 100, 1000) Representations Number Lines Base 10 Base 2 Two Factors One Factor Relationship between addends Properties and Relationships Relationship between addends and subtracts Commutative property Associative property Identity Relationships between models, equations and contexts Compositional Concepts/Properties Number Lines Base 10 Base 2 Two or more addends Compositional 10 Relationship between addends 	<ul style="list-style-type: none"> Meanings for Subtraction Difference Remainder Division Missing addend Single step Multiplication



The progression is not evaluative

Movement along the progression is not linear

Place value understanding
Number composition
Unitizing
Subitizing



Addition and Subtraction Progressions



OGAP Additive Reasoning Progression – Addition

As students learn new concepts or interact with new problem situations and problems structures, they may move up and down along this progression.

Additive	Additive Strategies			
Additive	Traditional US algorithm $\begin{array}{r} 18 \\ +27 \\ \hline 45 \end{array}$	Partial Sums $\begin{array}{r} 18 \\ +27 \\ \hline 30 \\ +15 \\ \hline 45 \end{array}$	Uses properties Associative Property $8+7+2=(8+2)+7=10+7$ Commutative Property $6+7=7+6$	Fact Fluency Fact Recall $6+7=13$ Derived Facts $6+6=12$ $6+7=13$
	Transparent algorithms Decomposes by place value and adds $18+27=?$ $18=10+8$ $27=20+7$ $30+15=45$ $18+20=38$ $38+7=45$	Flexible compensation $18+27$ $20+25$ 45	Using 10s $6+7$ $6+4=10$ and $10+3=13$	

Transitional (Tens)	Transitional Strategies	
Transitional (Tens)	Efficient Use of a Model <p>Jumps by multiples of 10 on a number line</p>	Inefficient decomposition to derive facts <p>"I know 5+5 is 10 and 3 more is 13"</p>
	Early Transitional Strategies Adding inefficiently with or without a model Adding on by tens $18+10=28$ $28+10=38$ $38+2=40$ $40+5=45$	Unitizes on a model <p>Makes jumps of 10 on a number line Combines or counts by 10s using base 10 representations</p>

Counting (Ones)	Counting Strategies	
Counting (Ones)	Mental counting strategies: <ul style="list-style-type: none"> Count on from first Count on from larger 	Counting on with physical or visual model. 6+7
	Early Counting Strategies Direct modeling and counting from 1 with concrete model (count 3 times) <p>"1,2,3,4,5,6,7,8,9,10,11,12,13"</p>	Underlying Issues/Errors <ul style="list-style-type: none"> Does not consider reasonableness of solution Error in counting, calculation, place value, property, equation, or model Units inconsistent or missing

Subitizing, Unitizing, Commutativity, Number Composition, Place value understanding

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OGAP Additive Reasoning Progression – Subtraction

As students learn new concepts or interact with new problem situations and problems structures, they may move up and down along this progression.

Additive	Additive Strategies		
Additive	Traditional US algorithm $\begin{array}{r} 73 \\ -38 \\ \hline 35 \end{array}$	Adding Up Efficiently $38+2=40$ $40+33=73$ $2+33=35$	Constant Difference: $73-38$ $+2+2$ $75-40=35$
	Transparent algorithms Decomposes by place value and subtracts $73-38=?$ $73=70+3$ $38=30+8$ $40-5=35$	Partial Difference: 73 -38 $40-5=35$	Fact Fluency Fact Recall Derived Facts Inverse relationship between addition and subtraction "7-2 is 5 because 5+2 is 7" Using 10s: 13-8 "13-3=10 and 10-5=5"

Transitional (Tens)	Transitional Strategies	
Transitional (Tens)	Efficient Use of a Model <p>Distance on number line</p>	Jump back on number line <p>13-8</p>
	Early Transitional Strategies Subtracting inefficiently with or without a model Using Base 10 models $73-38=35$	Adding up by tens $73-38=7$ $38+2=40$ $40+10=50$ $50+10=60$ $60+10=70$ $70+3=73$ $2+10+10+10+3=35$

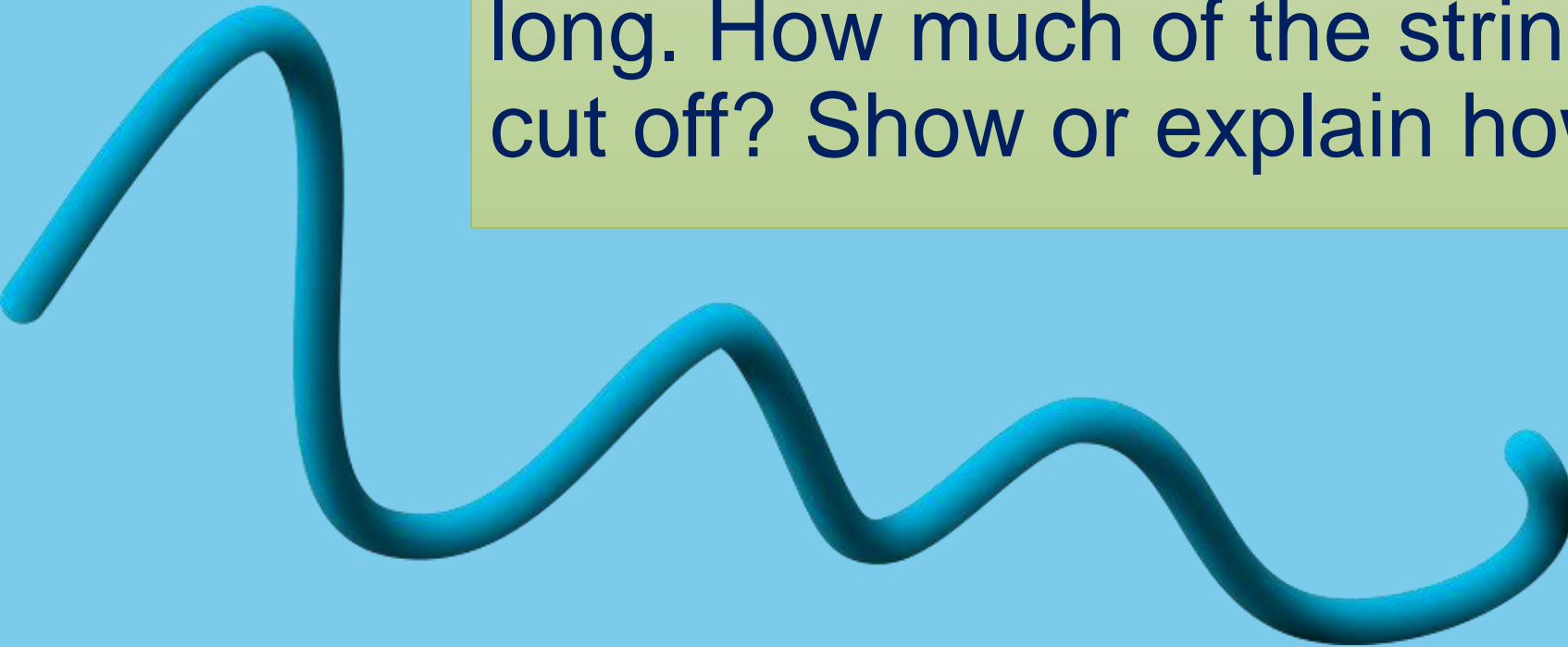
Counting (Ones)	Counting Strategies	
Counting (Ones)	Counting back (7-5) "7...6,5,4,3,2" Counting up "5...6,7"	Counting back (7-5) "7...6,5...that's 2" <p>Jumps by ones on number line $17-11=6$</p>
	Early Counting Strategies Direct modeling—separating from, separating to, or matching (counts 3 times) <p>8-5=3</p>	Underlying Issues/Errors <ul style="list-style-type: none"> Does not consider reasonableness of solution Error in counting, calculation, place value, property, equation, or model

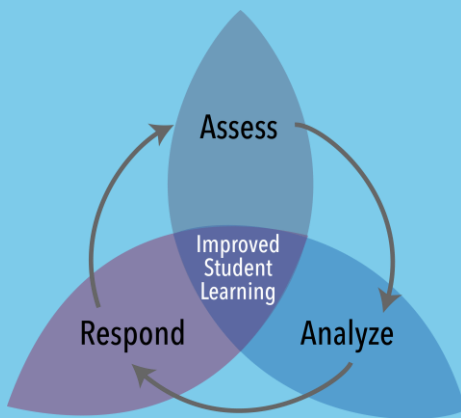
Subitizing, Unitizing, Commutativity, Number Composition, Base 10 understanding

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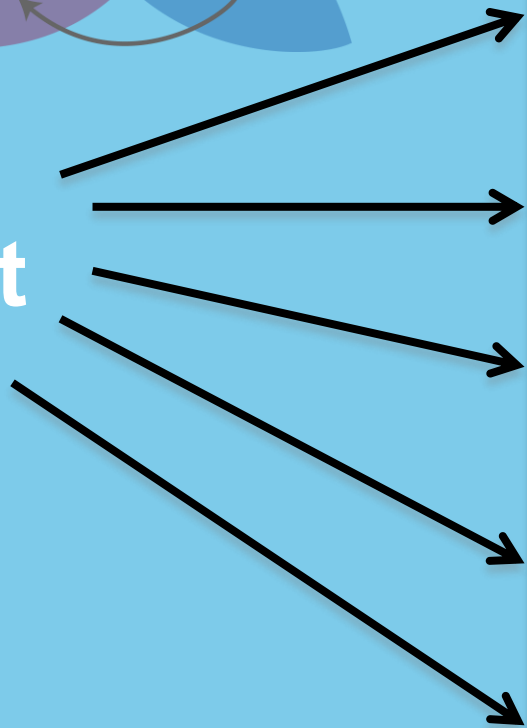
Solve two different ways.

A string was 56 inches long. Dylan cut some off. Now the string is 27 inches long. How much of the string did Dylan cut off? Show or explain how you know.





Sort



OGAP Additive Reasoning Progression – Addition

Additive

Additive Strategies

Traditional US algorithm	Partial Sums	Uses properties	Fact Fluency
$\begin{array}{r} 18 \\ +27 \\ \hline 45 \end{array}$	$\begin{array}{r} 18 \\ +27 \\ \hline 30 \\ +15 \\ \hline 45 \end{array}$	<ul style="list-style-type: none"> Associative Property: $8+7+2=(8+2)+7=10+7$ Commutative Property: $6+7=7+6$ 	<ul style="list-style-type: none"> Fact Recall: $6+7=13$ Derived Facts: $6+6=12$, $6+7=13$ Using 10s: $6+4=10$ and $10+3=13$

Transparent algorithms
Decomposes by place value and adds

$18 + 27 = ?$

$18 = 10 + 8$ $18 + 20 = 38$
 $27 = 20 + 7$ $38 + 7 = 45$
 $30 + 15 = 45$

Transitional Strategies
Efficient Use of a Model

$$\begin{array}{r} 18 + 27 \\ 10 + 8 \quad 20 + 7 \\ \hline 30 + 15 \\ \hline 45 \end{array}$$

Transitional (Tens)

Early Transitional Strategies
Adding inefficiently with or without a model

Adding on by tens

$18 + 10 = 28$
 $28 + 10 = 38$
 $38 + 2 = 40$
 $40 + 5 = 45$

Makes jumps of 10 on a number line

Combines or counts by 10s using base 10 representations

Counting Strategies

Mental counting strategies:

- Count on from first
- Count on from larger

Counting on with physical or visual model: $6+7$

Early Counting Strategies

Direct modeling and counting from 1 with concrete model (count 3 times)

Non-Additive Strategies

- Uses incorrect operation
- Models problem situation incorrectly
- Guesses
- Not enough information
- Uses procedures incorrectly

Underlying Issues/Errors

- Does not consider reasonableness of solution
- Error in counting, calculation, place value, property, equation, or model
- Units inconsistent or missing

OGAP Additive Reasoning Progression – Subtraction

Additive

Additive Strategies

Traditional US algorithm	Adding Up Efficiently	Constant Difference:	Fact Fluency
$\begin{array}{r} 73 \\ -38 \\ \hline 35 \end{array}$	$\begin{array}{r} 38 + 2 = 40 \\ 40 + 33 = 73 \\ 2 + 33 = 35 \end{array}$	$73 - 38 = 75 - 40 = 35$	<ul style="list-style-type: none"> Fact Recall Derived Facts Inverse relationship between addition and subtraction: $*7 - 2$ is 5 because $5 + 2 = 7$ Using 10s: $13 - 8$, $*13 - 3 = 10$ and $10 - 5 = 5$

Transparent algorithms
Decomposes by place value and subtracts

$73 - 38 = ?$

$73 = 70 + 3$ $73 - 30 = 43$
 $38 = 30 + 8$ $43 - 3 = 40$
 $40 - 5 = 35$ $40 - 5 = 35$

Transitional Strategies
Efficient Use of a Model

Transitional (Tens)

Early Transitional Strategies
Subtracting inefficiently with or without a model

Using Base 10 models

Jumps by tens on number line

Adding up by tens

$73 - 38 = ?$
 $38 + 2 = 40$
 $40 + 10 = 50$
 $50 + 10 = 60$
 $60 + 10 = 70$
 $70 + 3 = 73$
 $2 + 10 + 10 + 10 + 3 = 35$

Counting Strategies

Counting back (7 - 5)
 $*7...6, 5, 4, 3, 2$

Counting up
 $*5...6, 7$

Counting back (7 - 5)
 $*7...6, 5...that's 2$

Jumps by ones on number line

Counting back (7 - 5)
 $11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 17$
 $17 - 11 = 6$

Early Counting Strategies

Direct modeling—separating from, separating to, or matching (counts 3 times)

Non-Additive Strategies

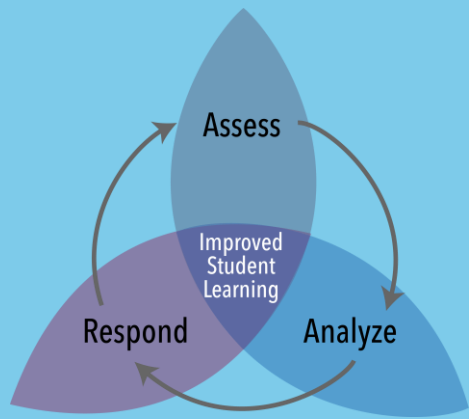
- Uses incorrect operation
- Models problem situation incorrectly
- Guesses
- Not enough information
- Uses procedures incorrectly

Underlying Issues/Errors

- Does not consider reasonableness of solution
- Error in counting, calculation, place value, property, equation, or model

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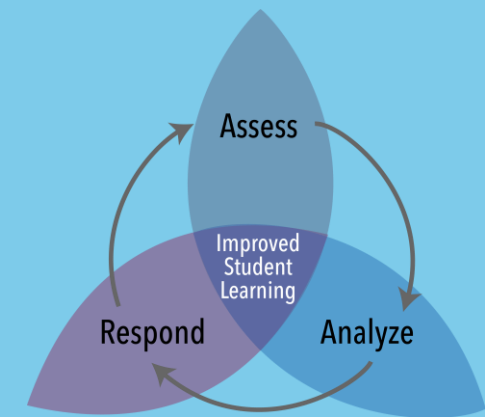


Questions to ask when analyzing student work:

- » What is the evidence of developing understanding that can be built upon?
- » What are issues or concerns that are evidenced in student work?
- » What are the next instructional steps based on that evidence?

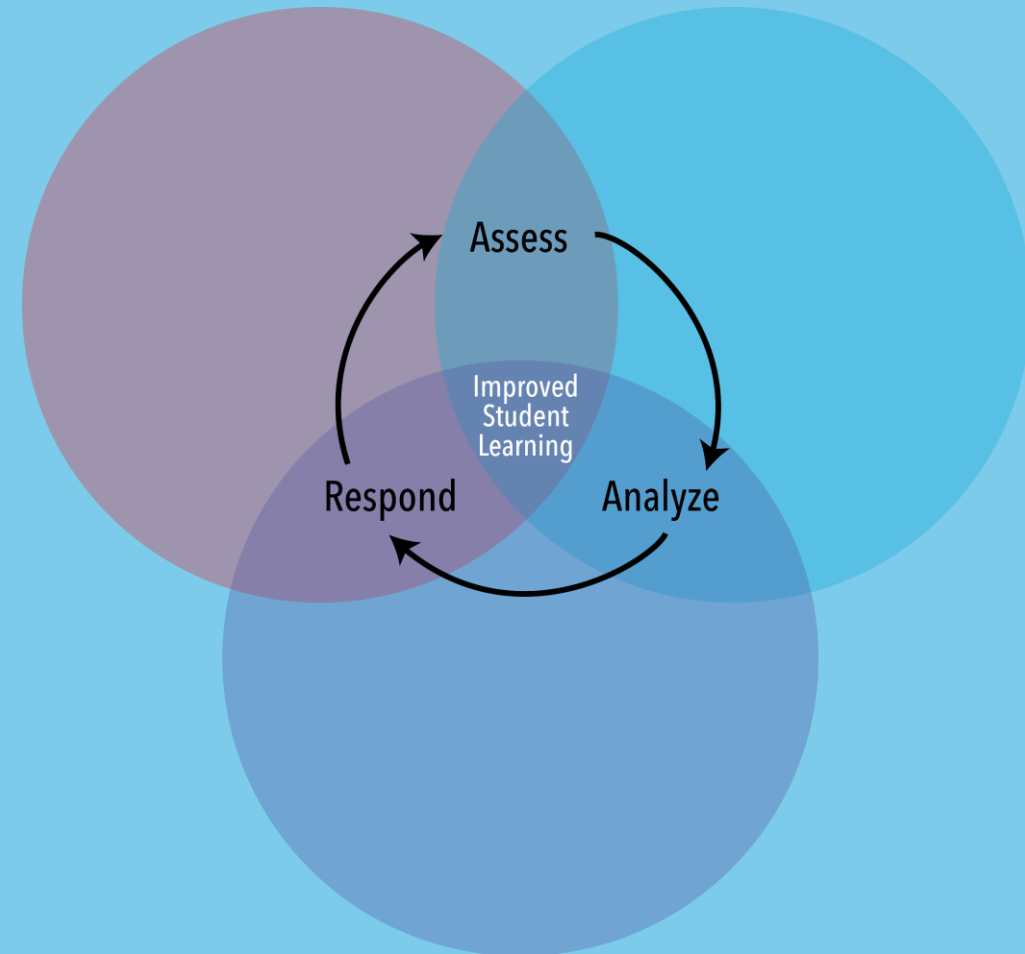
Select and Sequence

- Choose 2 pieces of student work that you could project side-by-side to start the next lesson.
- Think about how to help all students move forward on the progression
- What questions would you ask students to structure the discussion?



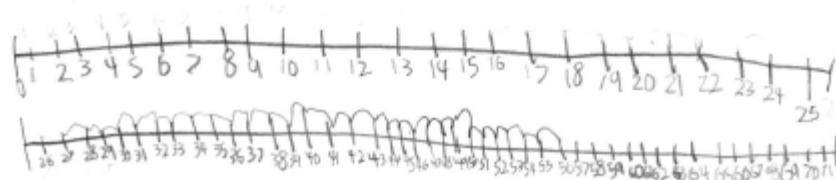
Formative Assessment Cycle

The essence of formative assessment is the relentless attention to evidence of student understanding and intentional and systematic use of the evidence for planning and instruction. (Popham, 2012)



What do these two students understand about number lines? About number?

A string was 56 inches long. Dylan cut some off. Now the string is 27 inches long. How much of the string did Dylan cut off?
Show how you know. **B**



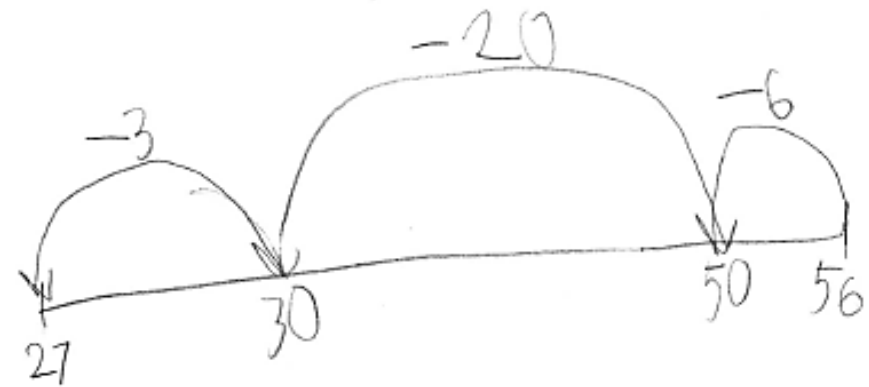
29, because I use a number line and hop backwards till 27 and then count the hops

A string was 56 inches long. Dylan cut some off. Now the string is 27 inches long. How much of the string did Dylan cut off?
Show how you know. **E**

$$56 - \boxed{29} = 27$$

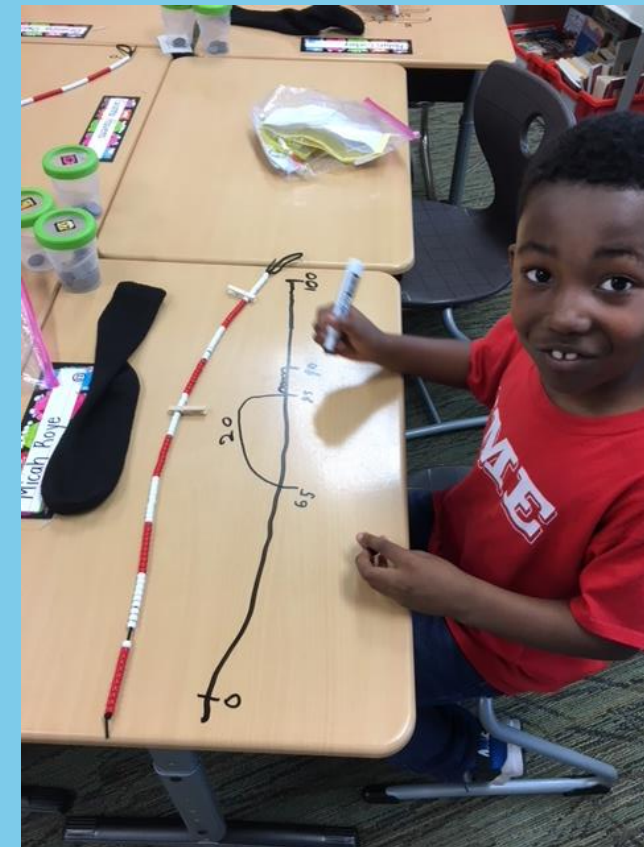
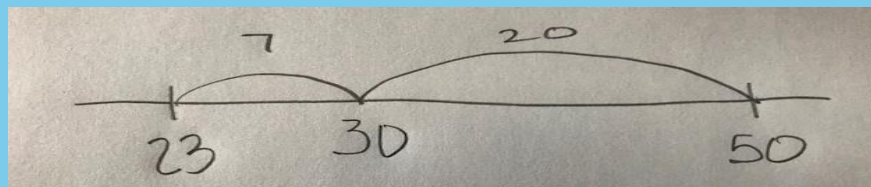
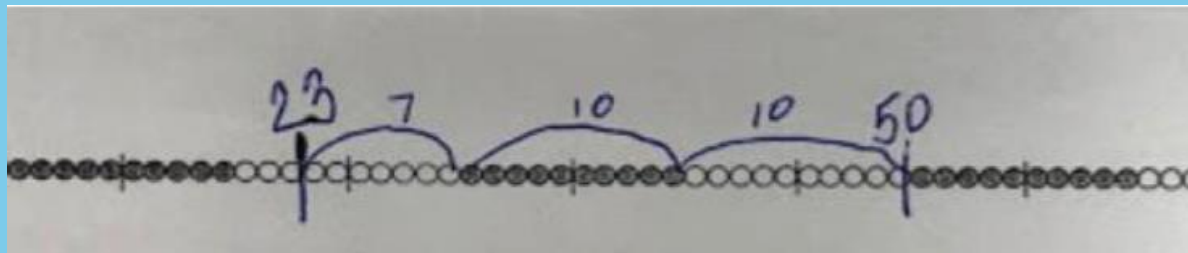
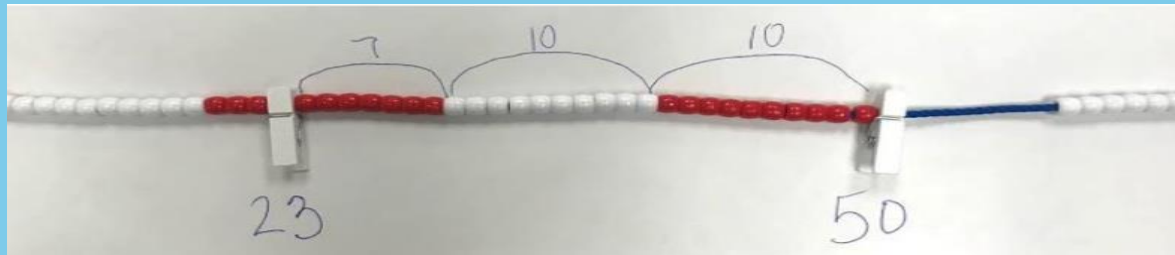
String

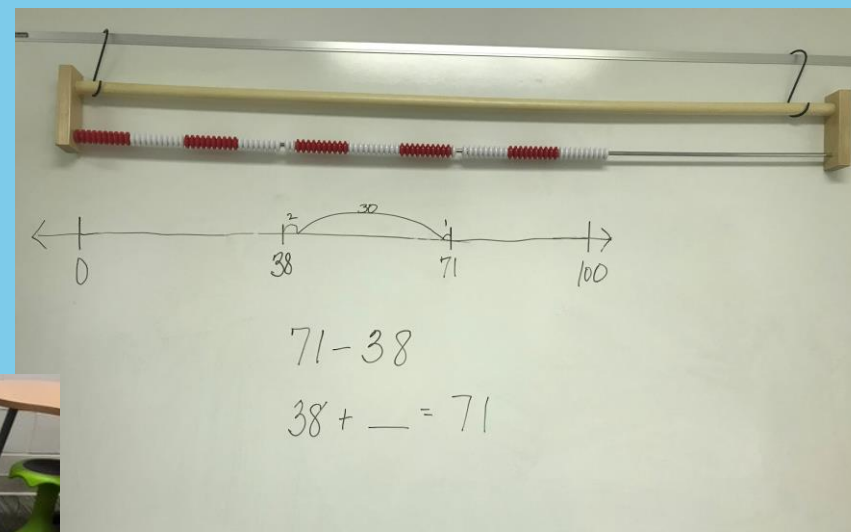
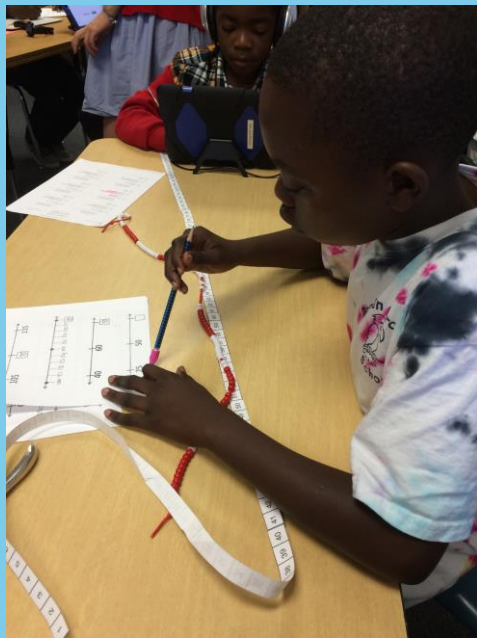
$$7 \times 4 + 1 = 29$$



Using Number Lines to Add and Subtract

$$50 - 23 = \underline{\quad}$$





For more information



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VIEW



| DISCUSS



| VOTE



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NCTM Research Conference

Tuesday, 3:45 – 4:15 PM

- *Revealing Base Ten Understanding Through Written Formative Assessment Tasks*

Wednesday, 3:00 – 4:00 PM

- *Improving Mathematics Through Formative Assessment.*

Research findings from a 2-year large scale OGAP impact and implementation study and ongoing OGAP implementation and support research.