OGAP

ONGOING ASSESSMENT PROJECT

Additive Reasoning

$$\begin{array}{r} 63 \\ -18 \\ \hline \end{array}$$
$63-10=53$
$53-3=50$
$50-5=45$

$$\begin{array}{r} 63 \\ -18 \\ \hline \end{array}$$
$2+40+3$
45

Session 1 – What is Number Sense
Session 2 – Counting
Session 3 – Subitizing
Session 4 – Number Composition
Session 5 – Number Lines
Session 6 – Addition
Session 7 – Subtraction
Session 8 – CCSSM (Addition/subtraction)
Session 9 – Problem Posing
Session 10 – Equality and Properties
Session 11 – Basic Fact Fluency
Session 12 – Item Bank

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Personal Parking Lot

Questions, Big Ideas, and Other Thoughts

1) What is cardinality? Give examples of cardinality. Why is it so important in early number understanding?

2) What is subitizing? Why can children use it for comparing small collections of objects yet are not able to use similar, related skills with larger numbers?

3) How is hierarchical inclusion related to addition and subtraction? Give specific examples.

4) What is compensation? How can you extend understanding of compensation to larger numbers and the operation of addition? Use specific examples.

5) Why is the strategy of “counting on” so difficult? What other skills must be integrated in order to understand “counting on”?

6) What is unitizing? Why is it important? Give specific examples (including those described and others you can imagine) that require a student to unitize.

7) Explain conservation in general and number conservation specifically. When do students use conservation of number? How can a teacher check for conservation of number and what behaviors indicate a lack of number conservation? Explain how conservation of number affects other important number concepts.

Cardinality is one of the four components of what is called "the number core" and therefore at the foundation for learning mathematics. The other mathematical aspects of the number core are the ordered number word list, 1-to-1 counting and written number symbols. While children first learn these aspects of number separately, they develop connections between them when they engage in meaningfully counting activities.

Cardinality answers the question "how many?" and refers to an understanding that when counting, the last number word said refers to how many things there are in the whole set. When we count a set of items, at the end of the counting action we make a mental shift from thinking of the last counted word as referring to the last counted thing to thinking of that word as referring to all the things in the set. For example, when counting 6 pencils, "1, 2, 3, 4, 5, 6," the 6 refers to the one last pencil you count when you say 6. Then you must shift to think of the 6 as representing all of the pencils. This is a major conceptual milestone for young children.

When asked "how many are there?" after counting a set, sometimes young children will go back and recount, indicating that they consider the answer to be the counting sequence itself (1, 2, 3, 4, 5, 6) rather than the last number said (6). This is evidence that the student does not yet possess an understanding of cardinality.

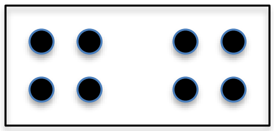
When children discover and understand cardinality they tend to generalize and apply the concept to all counts no matter the size of the set. However, even when students understand the importance of the last counted word, they don't necessarily understand cardinality (Fuson, 1998). Some students may understand that the last word said answers the question "how many" without grasping the more abstract idea of cardinality. In such cases they can say 6 to answer how many pencils in the set but when asked to show the 6 pencils they point to the last pencil and not the entire set.

National Research Council. (2009). Mathematics learning in early childhood: Paths toward excellence and equity. National Academies Press.

Subitizing is recognizing the numerosity of a group quickly and visually and connecting the quantity to the number name. People can see very small collections or groups and almost instantly tell how many there are without having to count them. Research shows that this ability develops in very young children, that it is related to the development of cardinality, and that it can be supported and extended by providing students with experiences where they focus on the numerosity and labeling of visual patterns with number words.

When you "just see" how many objects there are in a small set then you are using *perceptual subitizing*. For example, seeing apples on the table and saying there are 4 indicates you are perceiving the 4 apples intuitively and simultaneously. You can also determine the total of larger set (beyond the limits of perceptual subitizing) by breaking the group into smaller subgroups. For example, in the case of 8 apples you may see that there is a group of 5 and a group of 3 and very quickly, and often not consciously, say there are 8 apples. This is called *conceptual subitizing*.

In choosing visual patterns for subitizing activities, it is important to consider different types of things and different types of patterns that can be quickly recognized. Creating and using easily recognizable patterns, such the dot patterns found on dice, can help support conceptual subitizing and lead to the development many number and arithmetic concepts and strategies. For example, in the pattern below a student might say, “I know there are 8 because I saw 4 and another 4.” This is the foundation for understanding that 4 + 4 = 8 or 2 x 4 = 8.



Children who do not have enough experiences to support the development of conceptual subitizing are at a disadvantage when learning many number and arithmetic processes. Conceptual subitizing can be a stepping stone to adding and subtracting small numbers and supports more advanced methods for adding, subtracting, multiplying and dividing larger numbers. The ideas and skills associated with subitizing (cardinality, comparison, and addition) start developing very early but must be supported and built upon with intentional instruction. Developed well, these foundational concepts can be related and connected to provide strong building blocks of math. More complex ideas such as multiplication and fraction concepts can also be supported with subitizing activities.

Clements, D. H., & Sarama, J. (2014). Learning and teaching early math: The learning trajectories approach. Routledge

Hierarchical Inclusion is an understanding that a number contains all the previous numbers in the counting sequence. It is the understanding that numbers are nested inside each other and that the quantity grows by one with each count. For example, 19 is inside of 20, or 20 is the same as 19 + 1. If you remove one from a set of 20, the number goes back to 19. This mathematical big idea requires a logical inference and an informal operation on the whole set. If one more is added to 5 to get 6, then one removed from 6 would make 5, or nested inside the 6 is a 5. The ability to maintain the whole and understand how the parts are related to the whole requires an operation. To understand that when we have a set of 10 objects and one is removed we have 9 requires logical, mathematical reasoning. This understanding is necessary for counting on or back.

The big idea of hierarchical inclusion gets extended into a more complete understanding of number and the construction of two more big ideas: compensation and part/whole understanding. When students understand hierarchical inclusion, they can see the number as a unit while at the same time seeing it as being made up of parts. For example, they can consider that if 6 + 1 = 7 then 5 + 2 = 7 (nested inside the 7 is 5 and 2 more).

Fosnot, C. T., & Dolk, M. L. A. M. (2001). Young mathematicians at work. Portsmouth, NH: Heinemann

Compensation
In addition, a compensation is made when one addend is changed to compensate for a change of the same magnitude in the other addend, resulting in no change to the sum. One way that students can discover this relationship is when considering all the combinations that make up 10. When the combinations are listed in order or shown with colored connecting cubes, e.g., 10 + 0 is the same as 9 + 1, 8 + 2, 7 + 3, etc., students can observe the pattern that one addend increases by one while the other addend decreases by one. This is an example of compensation by 1. Modeling this relationship with concrete materials and equations can be very powerful for building understanding of compensation; later, students can use this understanding to flexibly to add numbers. For example, a problem such as 84 + 68 can be changed to 82 + 70 or 80 + 72 through compensation. In these examples, the total has not been changed but the student has made a flexible adjustment of the addends. This strategy is often referred to as "make it friendly" in elementary classrooms.

Compensation is one of the most flexible and useful mental math strategies, as well as an effective strategy for deriving unknown math facts. In order for students to be able to understand and use this strategy with larger numbers they must have purposeful and concrete instruction with smaller numbers and an explicit link to the use of the strategy with numbers of all sizes.

J. Van de Walle, Elementary and Middle School Mathematics, 8th edition, 2012

Counting On from one number is a difficult strategy for children to construct because they have to be able to think about one of the quantities abstractly and negate their earlier strategy of always counting from the beginning (from one). Understanding why the strategy works depends on developing an understanding of cardinality and hierarchical inclusion. Conservation is also at play here. The student who solves 5 + 3 by counting out 5, counting out 3 and then going back to count the total starting from 1 is unsure that 5 has been conserved while they have been working on adding 3. Counting all requires a cardinal-to-count shift and then a count on from the first addend.

To determine if a student has developed understanding of counting on, ask the student to count a set and then cover it. Then place additional objects next to it and ask how many there are in all. Do they go back to try to count the first set again, or can they build from the cardinality of the previously counted set without actually seeing it?

Counting on can be thought of as an application of many other early number concepts. Understanding of both the part/whole relationship and compensation also helps students use counting on as a strategy to effectively to solve addition problems. A student who recognizes the relationship between 6 + 3 and 5 + 2 is able to coordinate the understanding of part/whole, counting on, and compensation, rather than having to start over from the beginning as if it is a completely new problem.

Generally, students learn to count on from the first number and then transition to counting on from the larger number. When making this shift they are applying their understanding of more/less and the commutative property (3 + 5 = 5 + 3) in order to use a more efficient counting strategy. Students who count on their fingers and do not have a counting on strategy often struggle to solve problems greater than 10 because they cannot model all three quantities.

Unitizing is the ability to see a collection of ones as both a group and a set of individual ones. Central to developing preliminary place value understanding is the understanding that ten can be represented and thought of as one group of ten or ten individual units. This is a significant shift in thinking for children.

For example, the number 26 can be represented as 26 units, 1 ten and 16 units, or 2 tens and 6 units. This extends into an understanding that one hundred could be represented as one group of 100, 10 groups of 10, or 100 individual units.

Kamii (1995) states that because these ideas require logico-mathematical thinking they cannot just be explained or transmitted to students. Children may be able to paraphrase these ideas back if the teacher explains them, but to really understand these concepts, they must infer them on their own. This requires reflecting on patterns and relationships, generalizing, and developing understanding of why the pattern happens.

Understanding of unitizing is KEY for the development of number sense; it is a "Big Idea" or critical building block in mathematics. Unitizing requires that children use the number sequence to count not only individual objects but also groups of objects - and to count them both simultaneously. For learners, this is a big shift in perspective. Children have just learned to count objects one by one. Unitizing a group of ten things as ONE thing - one group- requires them almost to negate their original ideas of number (Fosnot & Dolk, 2001).

When children are able to unitize, you will see a shift in their reasoning, perspective, logic, and understanding of mathematical relationships. Unitizing underlies the understanding of place value and multiplication and division by ten. For example, understanding that 10 tens is 100 leads to the understanding that 30 tens is 3 hundreds or 300. It is important skills for all operations.

Students as early as kindergarten need to have extensive work with groups of ten, base ten models, counting by tens, and shifting from counting by ones to counting by tens. These experiences should continue and expand in later grades to the grouping of hundreds, tens, and ones, etc.

Fosnot, C. T., & Dolk, M. L. A. M. (2001). Young mathematicians at work. Portsmouth, NH: Heinemann

Conservation of Number

Conservation refers to the ability to determine that a certain quantity will remain the same despite adjustment of the container, shape, or apparent size. This is a central idea in Piaget's work on the development and teaching of early number. Children are said to conserve number if they are aware that when two sets have been shown to be equivalent, either by one-to-one correspondence or by counting, this equivalence is maintained even when one of the sets is rearranged. Piaget concluded that children must grasp the principle of conservation of quantity before they can fully develop the concept of number.

When asked to compare two sets of objects arranged in rows, young children may consider their visual properties rather than count both rows to compare the quantities. Consequently, when one row is compacted or pushed together, children without conservation of number will say that the shorter row contains fewer objects. Another example of a child's lack of understanding of conservation can be seen when the child has to recount a collection of items after the items have been rearranged in plain view without removing any of them. A student who lacks conservation of number will often struggle with addition and subtraction because they must count all from one instead of counting on from one number.

Conservation of number is one of the hallmarks of what Piaget called the concrete operational stage, where children can use logical thought or operations while they interact with physical objects. It is at this stage that children can develop more sophisticated strategies for adding and subtracting numbers without having to count every object. Research shows that children who have conservation of number demonstrate greater fluency in separately timed addition and subtraction problems than non-conserving children (Wubeena, 2013). This research highlights the importance of logical-reversible thought, an element necessary to conserve, as being a critical component to a child's ability to use more efficient strategies and understand inverse operations (e.g., if $4 + 3=7$ then $7 – 3 = 4$).

Conservation is one of Piaget's developmental accomplishments in which the child understands that changing the form of a substance or object does not change its amount, overall volume, or mass. You can often see the lack of conservation of volume in children when there are, for example, several different sizes of juice on a table, and they chose the glass that is the tallest because they perceive the taller glass as having more juice inside of it (even though the tallest glass may also be the thinnest). All the glasses may have the same amount of juice in them, but children who haven't accomplished conservation will perceive the tall glass as being most full.

Thompson, Ian. Teaching and Learning Early Number, Open University Press. 2005 p 4-10, 156

1d – Big Ideas Notes

Cardinality	Subitizing
Hierarchical Inclusion	**Compensation**

1d – Big Ideas Notes

Counting On	Unitizing
Conservation	

1d – Big Ideas Notes

Counting Collections

Counting provides an important foundation for understanding of number, number sense, base-ten, addition, subtraction, multiplication and division (National Research Council, 2001). Elementary students are expected to understand, count to, and operate on numbers in the hundreds, but how often do we expect them to actually develop and apply their understanding to count large sets of items, particularly tens or hundreds of items at a time? In the Counting Collections activity, students count collections of markers, pencils, math manipulatives, buttons, bottle caps, dice, or any countable items, to answer the question “how many?” As students work in pairs to count, organize, keep track of, and record their results, they develop understanding of the complex concept of number and increasingly sophisticated strategies that make sense to them.

Counting Collections provides students with different levels of understanding meaningful opportunities to count large quantities of objects and develop collaborative work skills. They work together to solve a real problem: “How many are there in this set?” As students work together to determine the total amount in a collection, they develop and apply important foundational concepts: assigning one number to one object (one-to-one correspondence), counting on from one number, and using the last number in a count to represent the quantity (cardinality). They also gain important practice applying the count sequence, and in particular transitioning across decades (10s) or centuries (100s). As students look for and discover ways to organize and count more efficiently by groups, they use concepts of unitizing and conservation. For example, they might count out the single items one-by-one to make groups of tens and then count each group by tens (10, 20, 30, 40…), using their understanding that ten items are also one group of ten, no longer needing to count each object.

Engaging students in Counting Collections regularly throughout the year allows the teacher to formatively assess students’ developing understanding of number concepts and strategies and gives students repeated practice to develop more sophisticated counting strategies with varied collections as they learn from each other. There are different ways to structure and integrate the activity into your math curriculum. For example, the teacher might set aside 2-3 days in a row at the beginning, middle and end of the year or one day every other week for this activity. Counting Collections begins with a whole group launch, followed by students working in pairs to count collections, and then a whole class discussion or closing to highlight different strategies.

I. Planning for Counting Collections

Before introducing Counting Collections to students, there are several things to plan for ahead of time. Each pair of students will need one collection of items to count; these collections should be organized into bins, bags, or boxes to make them easy to transport and store. Keep in mind that larger items will be easier for young students to count, and tiny items or items that roll or stick together may be difficult to manage. You may also want to provide cups, plates, or bowls for students to organize their collection as they count. The difficulty of a collection can be varied by the size of the quantity, the variety of attributes (e.g., size, color, type), or the existence of composite units (i.e., items packaged in groups of tens or hundreds). Purposefully selecting what students will count provides opportunities for differentiation during this activity (i.e., students who are still learning early number concepts should be given smaller sets.) Consider spaces around the classroom where students can spread out to work with their partner—they will need more space than just a desk to count and organize larger collections. Pair students intentionally by considering their developing understanding and strategies. Teachers can have students complete a Counting Quantities (CQ) task from the OGAP Item Bank ahead of time (or after the first day) and sort the work in relation to the OGAP Number Progression to inform the pairing of students. For example, a teacher might pair a student who is showing 1-1 correspondence and keeps track but struggling to count past 30, with a student who has mastered the counting sequence but lacks organization.

Plan how you will introduce the activity and consider your goals. What do students need to know in order to be successful with this activity? What will you look for as students work? What strategies or mathematical concepts do you anticipate highlighting as you have students share their strategies? There are many ways to extend the activity for students who are successfully counting and recording. For example, you could ask students to estimate before counting, determine how many tens or equal groups are in their collection, or find the total of two collections.

II. Launching Counting Collections

Begin with a whole group launch to introduce the activity, establish expectations and routines, and/or provide a brief mini-lesson to guide students in their work. Introduce the activity by setting up a real problem for them to solve: e.g., "I need to know how many supplies we have so that I know if I need to order more" or "I need to take inventory of our classroom books." Children are naturally curious to answer real life questions like, "do we have enough?" Once a purpose or context is set, set clear expectations for how they will work. For example, you can model how to work with a partner to count a shared collection or decide on a strategy. Then, as students become familiar with the activity, you can introduce how to record or organize as they count.

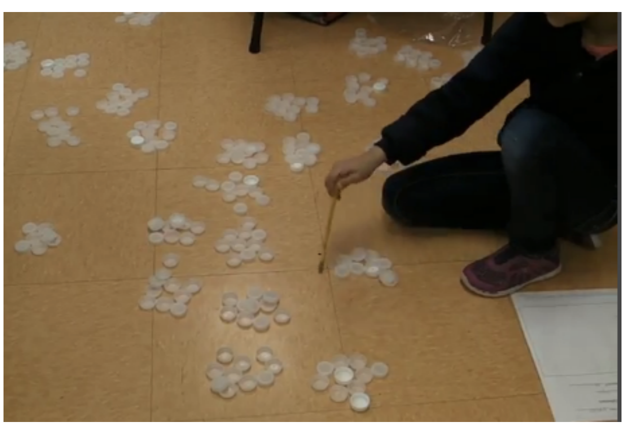
Make sure to tell students how to get their materials, who their partner will be, and where they will work, but refrain from telling them *how* to count their collections. Rather than directing them to specific strategies, ask them to share ideas. It is important to let them try out their own ideas and learn from their experiences. After doing the activity one or more times, focus on highlighting ideas shared on previous days of Counting Collections, allowing students to investigate new ways to count quantities as they work and through opportunities to learn from others.

In the excerpt below, the teacher asks students to think about strategies they used last time and focuses them on efficiency of strategy since the collections will be larger this time.

T: When we did counting collections before we talked about strategies that are efficient. This time your collections are bigger, so you really now need to think about strategies you are using when counting. Who can explain what efficient strategies are?
S: Counting by 5's or 10's
T: Ok, can you explain that a little more?
S: You take ten and then you keep making ten and count out those
T: Who can tell me why that would be an efficient strategy?
S: Because if you are doing something else like color, then you will have a lot of different numbers to add, and if you are doing by 5's or 10's you can just count by 10's
S: Because counting by ones you have to start over and you might miss some and get a different answer
T: We found that out with our first counting collections didn't we? Some people went back to check and see if their answer was right, and it usually wasn't the same answer when they were counting by ones.
S: And it's going to take longer.

III. During Counting Collections

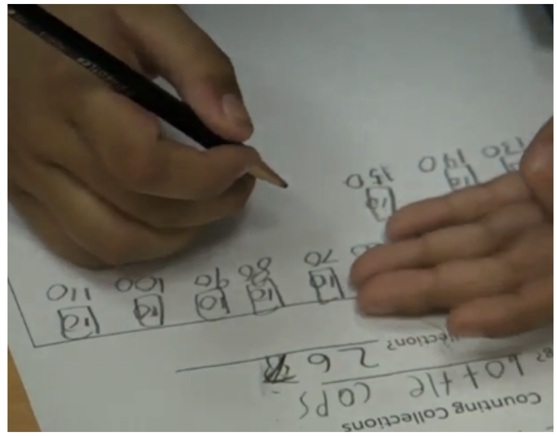
Children benefit from working together, talking to each other, and sharing their thinking (Shwerdtfeger & Chan, 2007). Once students have their collections, they will need to plan together and agree upon a counting strategy--they may decide to count each item one-by-one, or make groups of two, five, or ten. They will learn more about what they need to do by actually doing it. For example, if the collection has varied attributes, they may want to sort them by color, size, or type. Let them try this, and eventually they will discover that sorting by random sized groups requires combining totals and does not help them find a total efficiently.



After students begin to count, they will need to develop strategies for keeping track of the items they have counted, recording the quantities, and determining the total. As students work, circulate to listen in, ask questions, and collect evidence of their thinking. Begin by asking each group how they are counting and observe how they group or organize their collection. Other questions to ask include: How many do you have or how many do you have here? How do you know? Are you sure? Could you prove that to me? Could you count them a different way?

 This product is the result of a collaborative effort between the Ongoing Assessment Project (OGAP) and the Consortium for Policy Research in Education (CPRE) which was funded by the National Science Foundation (DRL-162088).

As students learn to organize the objects into groups, you can ask questions to help them think about the purpose of grouping and the relationship between the number of groups, the total quantity, and the written number:



- How are you grouping them?
- Could you group them a different way?
- How many do you have in all? Can you write the number?
- How many groups of ten do you have?
- Without counting your piles can you predict how many piles of ten you have? Ones you have?
- If you counted 386 items, how many piles of ten would you have? How do you know?

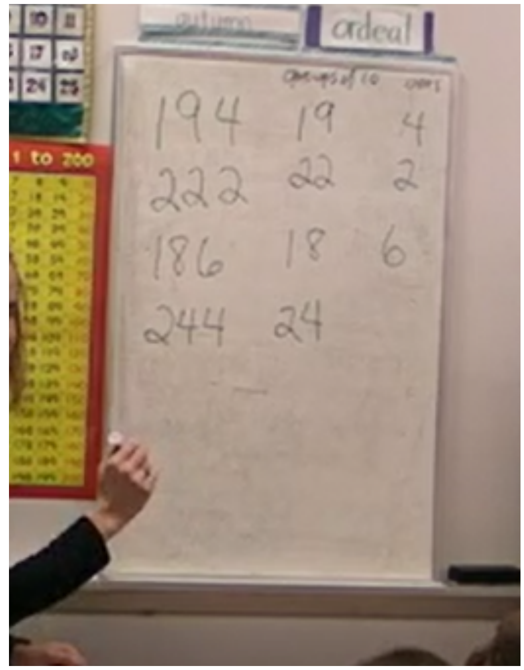
Asking questions while students are working can help you gather formative assessment information about student thinking and understanding. As you circulate, consider a discussion topic for the closing. What trends do you notice? What pairs of students will you ask to share? How will you sequence the sharing?

The OGAP Counting Collections Checklist is designed to help you keep track of your observations in relation to how children are organizing the count, keeping track of counted objects, determining the total, and any underlying issues and errors. Use this evidence to inform future instructional steps or to strategically partner students. Instructional responses might include:

- Identifying students that may need support with working together or counting and providing specific instruction when necessary.
- Identifying groups that others can briefly visit to gather new ideas for counting or organizing.
- Selecting and sequencing student strategies for the whole group discussion.
- Determining the focus of the mini-lesson for the next launch.

IV. Closing Counting Collections

Counting Collections closes with a whole group discussion where “strategies, coming straight from the children, become topics of group conversation” (Schwerdtfeger & Chan, 2007). The discussion can link back to the launch or feature new ideas and efficient strategies that may be helpful to others. Use pictures from the work time, or have specific groups leave their collections out so others can see what they have done. This is also a good time to highlight important mathematical concepts that arise in student conversation. For example, you can ask questions such as: How did you keep track of the items you counted? Did anyone tried to count in more than one way? Did you get the same answer? Why or why not? Did anyone make groups? How did this help you figure out the total? These discussions can then become the topic of conversation for the next launch.



In the picture shown here, the teacher had 2nd grade students share the total of their collections, along with how many groups of ten they made and how many ones were left over. After collecting three different examples, students began to see a pattern between the digits in the numeral and the groups of tens and ones. When one group shared that they had 244 and 24 groups of ten, she asked the class how many they thought were left over. When students were able to correctly “guess” 4 she asked them what patterns they were noticing and emphasized the place value of the numbers.

T: Can anybody tell me what you noticed? About the groups of ten that you made and the number? Turn and talk about that with your neighbor. What pattern do you notice?

S: I noticed that all the groups of ten are the numbers at the beginning of the number. So like 19 and 4 makes 194

T: So the 19 is 19 groups of ten and it’s also the first two digits of the total. Anybody else?

S: The first two numbers is going to be how many tens and the last number is going to tell you how many left over.

T: So if we made the number 568, how many groups of tens do you think we would make? How do you know? Turn and talk.

Although many students had made groups of ten to count their collections, they hadn’t seen this connection until the whole group sharing and discussion.

Counting Collections is a powerful instructional activity that offers access to important mathematics for a wide range of students within a classroom. The mathematical focus of a counting collection lesson can vary depending on the grade level, the time of year and the students. A teacher can use counting collections to build understanding of number concepts such as counting, cardinality, conservation, unitizing, and base ten understanding.

The *freedom within form* structure of the activity allows students to construct their own systems and strategies to count the collection and surfaces the depth of student understanding. This provides an excellent opportunity for formative assessment and allows the teacher to tailor the focus of the lesson to meet the needs of individual or small groups of students. Changing the size and variety of the collection, as well as the support structures, in response to differing students' needs creates a platform for differentiation throughout the lesson. Counting Collections creates multiple entry points and promotes development of increasingly sophisticated strategies that students can employ for counting, keeping track, and recording the quantity of a given collection.

References:

Shwerdtfeger, J. & Chan, A. (2007, March). Counting Collections. *Teaching Children Mathematics*, *13* (7), 356-361.

Teacher Education by Design (TEDD) (2017). Counting Collections. University of Washington. Retrieved from https://tedd.org/activities/counting-collections/

Counting Collections

Student	How do they count collections?					How do they keep track?				How do they organize as they count?				How do they organize to determine the total?			Notes
	1s	2s	5s	10s	Other	Did Not	Tag	Move	Group by	Did Not	IC	R	CG	Did Not	R	O	

IC – inconsistent, CG- Consistent Groups, R-Random, O-Organized

Questions to ask the students:

- How many do you have or how many do you have here**?**
- How do you know?
- Are you sure?
- Could you prove that to me?
- Could you count them a different way?
- How are you grouping them?
- Could you group them a different way?

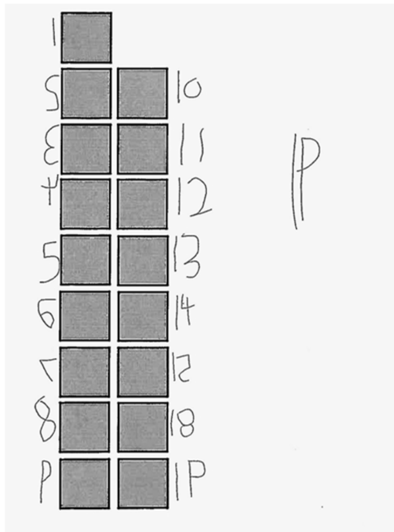
* **H**ow many groups of ten do you have?

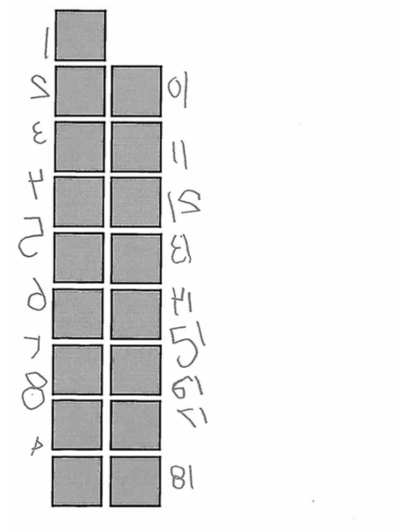
* How many do you have in all? Can you write the number?

* Without counting your piles can you predict how many piles of Ten you have? Ones you have?

* If you counted 386 items, how many piles of ten would you have? How do you know?

Omar built 2 towers. How many blocks did he use altogether? Show how you know.





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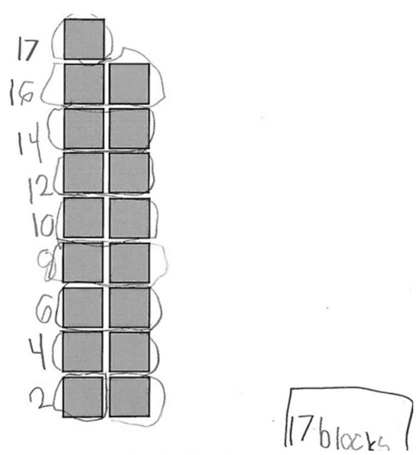

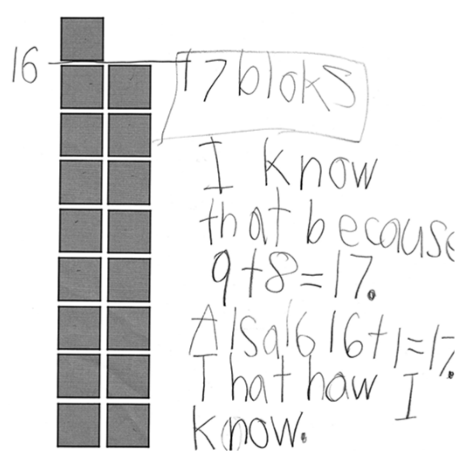

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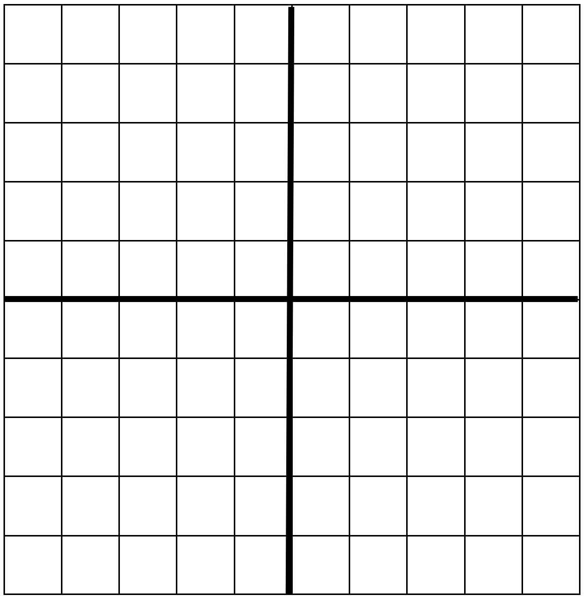
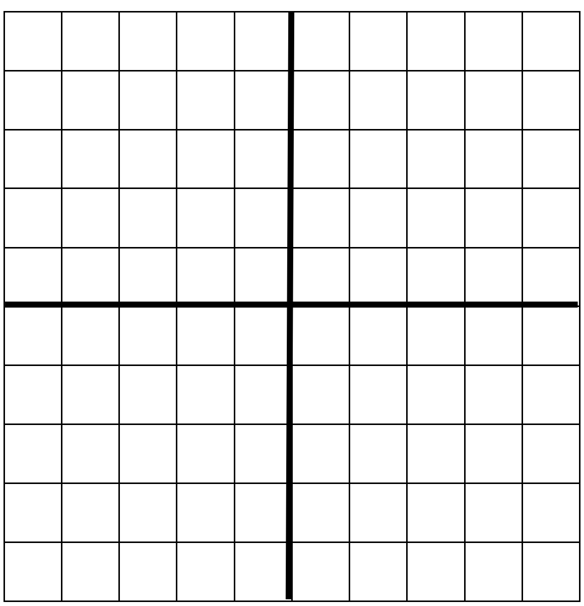
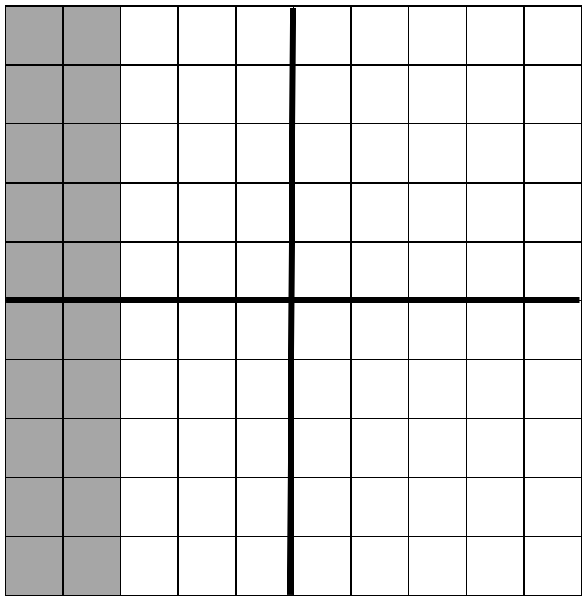
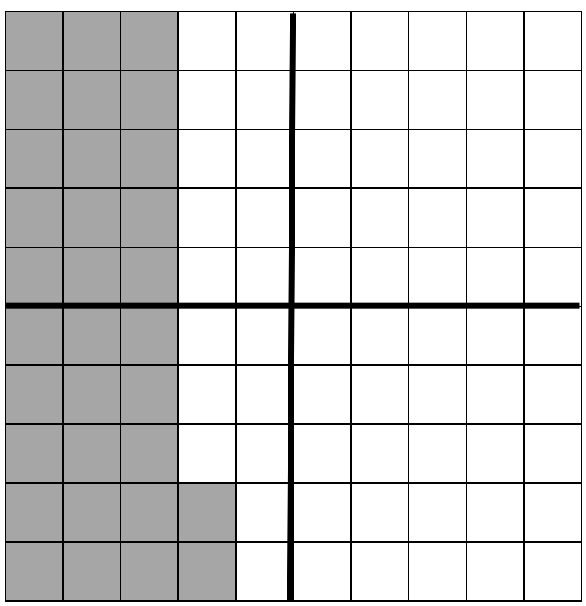
Hundred Grids

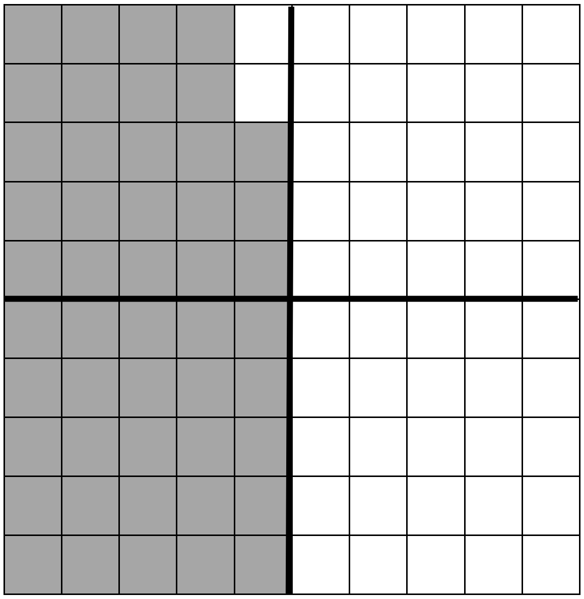


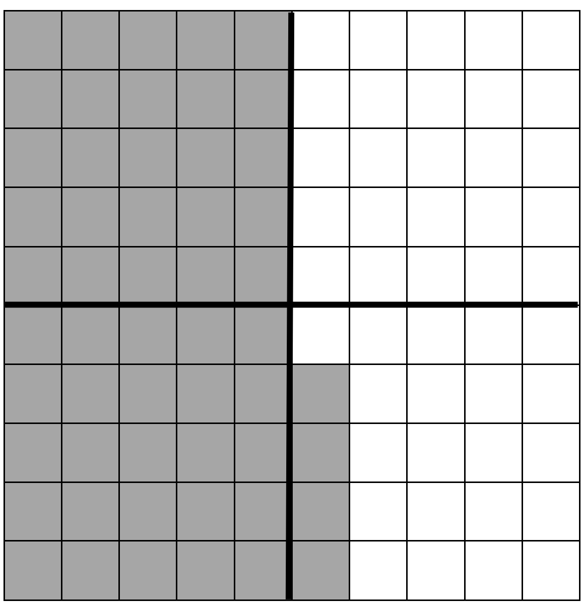


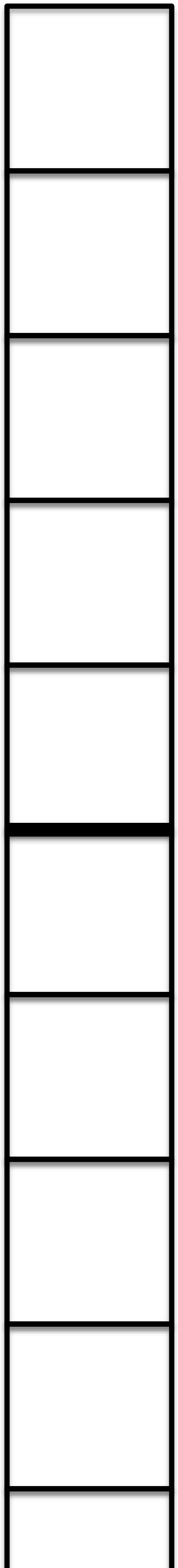
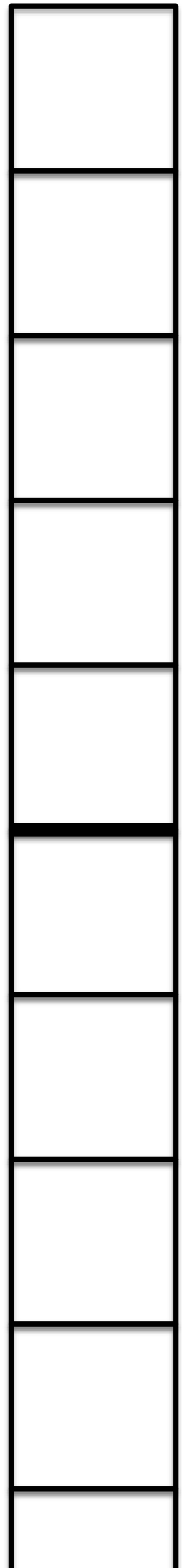


Hundred Grids







100

10

200

20

300

30

400 40

500 50

600 60

700 70

800 80

900 90

1000 1

2000 2

3000 3

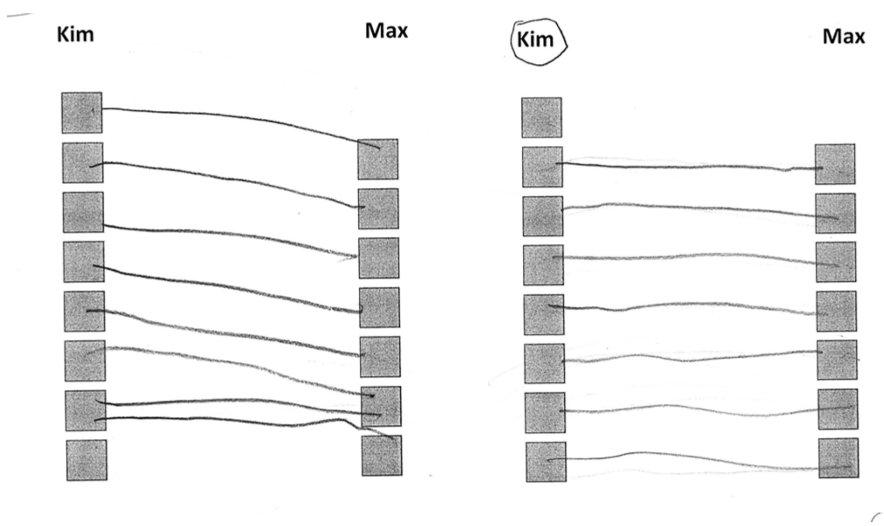
4000 4

5000 5

6000 6

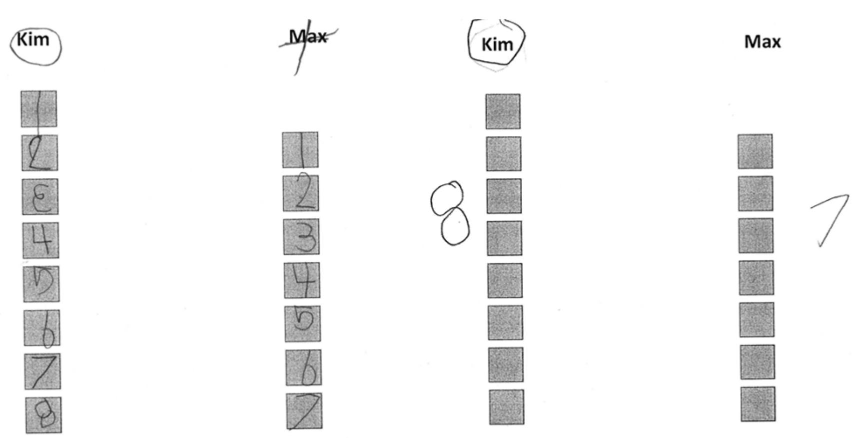
7 0 0 0	7
8 0 0 0	8
9 0 0 0	9

Who has more? Show how you know.



1

Who has more? Show how you know.



2

Cut around each box then fold the number line on the dotted line. Paste down or laminate. For larger number lines enlarge to A3.

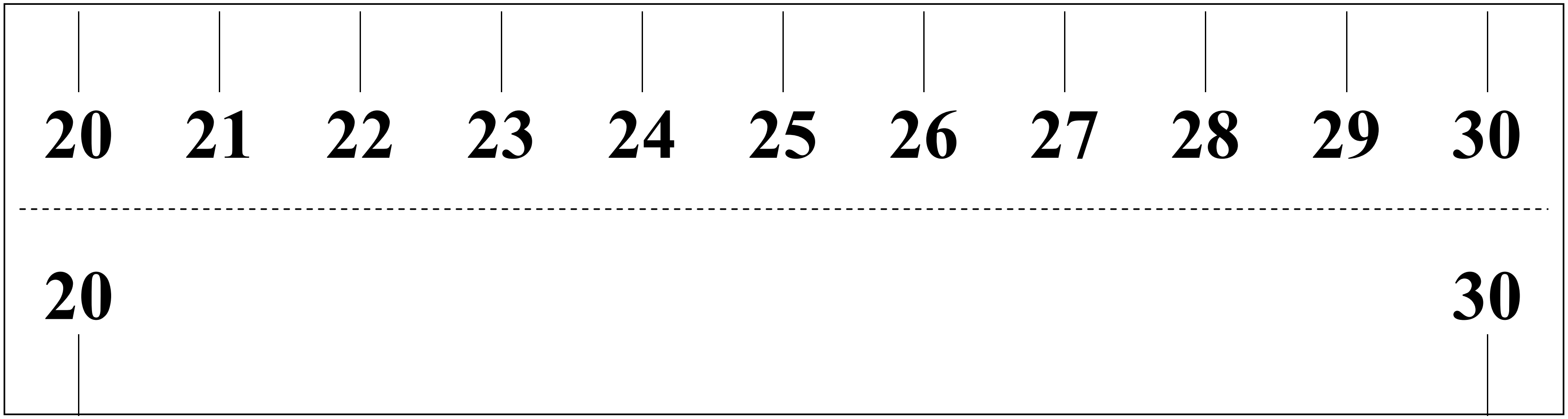
0 1 2 3 4 5 6 7 8 9 10

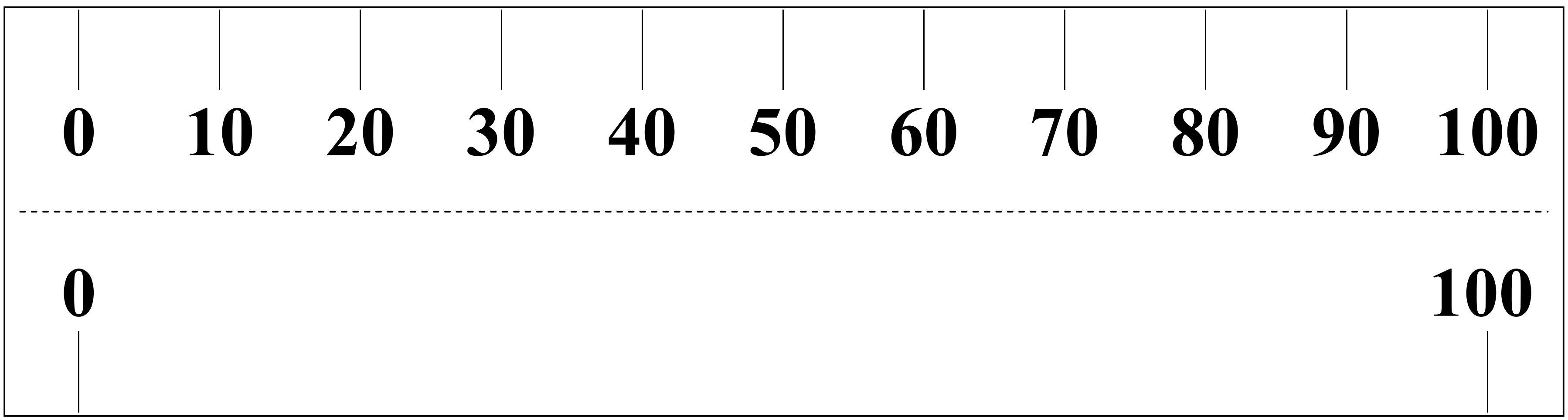
0 10

10 11 12 13 14 15 16 17 18 19 20

10 20

Cut around each box then fold the number line on the dotted line. Paste down or laminate. For larger number lines enlarge to A3.





Subtraction Research

1.) It is almost impossible to develop procedural fluency with multi-digit algorithms without strong base ten understanding. Students do not need to develop place value understanding before performing multi-digit computation. Rather, the two go hand in hand and should be developed simultaneously, and consistently used to deepen understanding of each other. (Adding it Up, NRC, 2000)

2.) When students fail to grasp the concepts that underlie procedures, they frequently generate flawed procedures that result in systematic errors, which teachers mistakenly recognize as “silly errors”. (Dossey et al, 2008)

3.) Understanding and fluency are related. Given conventional instruction that emphasizes practicing procedures, void of an ongoing revisiting of conceptual structures, a substantial percentage of students will not be successful in subtraction. (R. Ritchart, 2005)

4.) Subtraction is made more difficult by the impossibility of maintaining the model as a referent. For example, when solving 73 – 26 with base ten blocks, the 76 is represented with 7 tens blocks and 3 ones blocks. In order to take away 26, the model 73 must be changed, making the ability to refer to the original model and keep track of the steps to solve the problem, challenging for many students. (van de walle, 2010)

5.) Given time to develop meaning for a model and then connect a written procedure to it, students have shown high levels of performance using both written and mental procedures and the ability to fall back on the model to help explain their answers. (van de walle, 2010)

6.) Algorithms should grow out of physical models versus physical models being used to justify an algorithm. (van de walle, 2010)

7.) The act of inventing algorithms and comparing and contrasting them, is a kind of problem solving that has the greatest influence on understanding. Procedural fluency is built directly on understanding. Students who invent their own correct procedures have been proven to have strong procedural fluency. (Russell, Economopolous, Bastable, 2007)

8.) The best way to solve a problem is dependent on the numbers and the situation. *(Van de Walle, 2010)*

9.) Multi-digit subtraction is heavily influenced by instruction. Rich, effective instruction in subtraction is often hindered by teachers' inflexibility with multi-digit subtraction. (Adding It Up, NRC, 2000)

10.) A teacher's ability to teach subtraction is the second most common reason (flexible place value understanding is the first) students are tangled with multi-digit subtraction. While they are comfortable with their own ability to do subtraction with regrouping and describe the procedure, they are far less articulate with the conceptual underpinnings of subtraction…
(Ball, Shram, Fieman-Nemser, 2009)

Examine the following examples of student procedures for solving the same subtraction problem.

Answer these questions about each strategy:

1. What did the student do?
2. Why does it work?
3. What do you know about this student's understanding of number? Of subtraction?
4. Try the strategy with these numbers: 726 – 374
5. Would it work for all whole numbers?
6. When might it be a good strategy to use? A limiting strategy to use?

Student 1

63
−18

63 − 10 = 53
53 − 3 = 50
50 − 5 = 45

Student 2

$6\cancel{3}^{13}$
$-\cancel{1}^{2}8$

45

Student 3

63
−18

2 + 40 + 3

45

Student 4

55 8
$\cancel{6}\cancel{3}$
−18

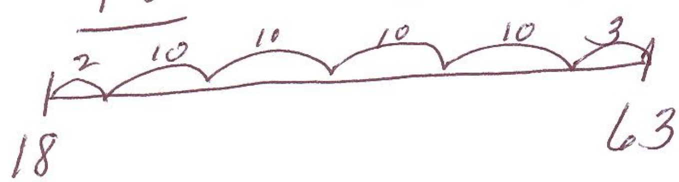
45

Student 5

$$\begin{array}{r} 63 \\ -18 \\ \hline \end{array} = \begin{array}{r} 65 \\ -20 \\ \hline 45 \end{array}$$

Student 6

$$\begin{array}{r} 63 \\ -18 \\ \hline \end{array} \quad 18 + \_\_\_ = 63$$



Student 7

$$\begin{array}{r} 63 \\ -18 \\ \hline \end{array}$$

$50 - 5 = 45$

Student 8

$$\begin{array}{r} 63 \\ -18 \\ \hline \end{array}$$

28, 38, 48, 58, 63

10 + 10 + 10 + 10 + 5

45

Compare and contrast the different strategies. Can you find students who have similar strategies? How are they similar?

Compare these strategies to the traditional algorithm.

8b- **Progression of Addition and Subtraction in the CCSS**

Grade	K	1	2	3
Fluent within…				
Addition and Subtraction within…				
Strategies and Algorithms				

 This product is the result of a collaborative effort between the Ongoing Assessment Project (OGAP) and the Consortium for Policy Research in Education (CPRE) which was funded by the National Science Foundation (DRL-16208888).

Mathematics | Kindergarten

In Kindergarten, instructional time should focus on two critical areas: (1) representing, relating, and operating on whole numbers, initially with sets of objects; (2) describing shapes and space. More learning time in Kindergarten should be devoted to number than to other topics.

(1) Students use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set; counting out a given number of objects; comparing sets or numerals; and modeling simple joining and separating situations with sets of objects, or eventually with equations such as $5 + 2 = 7$ and $7 - 2 = 5$. (Kindergarten students should see addition and subtraction equations, and student writing of equations in kindergarten is encouraged, but it is not required.) Students choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away.

(2) Students describe their physical world using geometric ideas (e.g., shape, orientation, spatial relations) and vocabulary. They identify, name, and describe basic two-dimensional shapes, such as squares, triangles, circles, rectangles, and hexagons, presented in a variety of ways (e.g., with different sizes and orientations), as well as three-dimensional shapes such as cubes, cones, cylinders, and spheres. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.

Grade K Overview

Counting and Cardinality

- **Know number names and the count sequence.**
- **Count to tell the number of objects.**
- **Compare numbers.**

Operations and Algebraic Thinking

- **Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.**

Number and Operations in Base Ten

- **Work with numbers 11–19 to gain foundations for place value.**

Measurement and Data

- **Describe and compare measurable attributes.**
- **Classify objects and count the number of objects in categories.**

Geometry

- **Identify and describe shapes.**
- **Analyze, compare, create, and compose shapes.**

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Counting and Cardinality K.CC

Know number names and the count sequence.

1. Count to 100 by ones and by tens.
2. Count forward beginning from a given number within the known sequence (instead of having to begin at 1).
3. Write numbers from 0 to 20. Represent a number of objects with a written numeral 0-20 (with 0 representing a count of no objects).

Count to tell the number of objects.

4. Understand the relationship between numbers and quantities; connect counting to cardinality.
   a. When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.
   b. Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.
   c. Understand that each successive number name refers to a quantity that is one larger.
5. Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

Compare numbers.

6. Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.[1]
7. Compare two numbers between 1 and 10 presented as written numerals.

Operations and Algebraic Thinking K.OA

Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

1. Represent addition and subtraction with objects, fingers, mental images, drawings[2], sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.
2. Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.
3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1).
4. For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.
5. Fluently add and subtract within 5.

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[1]Include groups with up to ten objects.
[2]Drawings need not show details, but should show the mathematics in the problem. (This applies wherever drawings are mentioned in the Standards.)

Number and Operations in Base Ten	K.NBT

Work with numbers 11–19 to gain foundations for place value.

1. Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

Measurement and Data	K.MD

Describe and compare measurable attributes.

1. Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.
2. Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.*

Classify objects and count the number of objects in each category.

3. Classify objects into given categories; count the numbers of objects in each category and sort the categories by count.[3]

Geometry	K.G

Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).

1. Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*.
2. Correctly name shapes regardless of their orientations or overall size.
3. Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").

Analyze, compare, create, and compose shapes.

4. Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).
5. Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.
6. Compose simple shapes to form larger shapes. *For example, "Can you join these two triangles with full sides touching to make a rectangle?"*

[3]Limit category counts to be less than or equal to 10.

Mathematics | Grade 1

In Grade 1, instructional time should focus on four critical areas: (1) developing understanding of addition, subtraction, and strategies for addition and subtraction within 20; (2) developing understanding of whole number relationships and place value, including grouping in tens and ones; (3) developing understanding of linear measurement and measuring lengths as iterating length units; and (4) reasoning about attributes of, and composing and decomposing geometric shapes.

(1) Students develop strategies for adding and subtracting whole numbers based on their prior work with small numbers. They use a variety of models, including discrete objects and length-based models (e.g., cubes connected to form lengths), to model add-to, take-from, put-together, take-apart, and compare situations to develop meaning for the operations of addition and subtraction, and to develop strategies to solve arithmetic problems with these operations. Students understand connections between counting and addition and subtraction (e.g., adding two is the same as counting on two). They use properties of addition to add whole numbers and to create and use increasingly sophisticated strategies based on these properties (e.g., "making tens") to solve addition and subtraction problems within 20. By comparing a variety of solution strategies, children build their understanding of the relationship between addition and subtraction.

(2) Students develop, discuss, and use efficient, accurate, and generalizable methods to add within 100 and subtract multiples of 10. They compare whole numbers (at least to 100) to develop understanding of and solve problems involving their relative sizes. They think of whole numbers between 10 and 100 in terms of tens and ones (especially recognizing the numbers 11 to 19 as composed of a ten and some ones). Through activities that build number sense, they understand the order of the counting numbers and their relative magnitudes.

(3) Students develop an understanding of the meaning and processes of measurement, including underlying concepts such as iterating (the mental activity of building up the length of an object with equal-sized units) and the transitivity principle for indirect measurement.[1]

(4) Students compose and decompose plane or solid figures (e.g., put two triangles together to make a quadrilateral) and build understanding of part-whole relationships as well as the properties of the original and composite shapes. As they combine shapes, they recognize them from different perspectives and orientations, describe their geometric attributes, and determine how they are alike and different, to develop the background for measurement and for initial understandings of properties such as congruence and symmetry.

[1]Students should apply the principle of transitivity of measurement to make indirect comparisons, but they need not use this technical term.

Grade 1 Overview

Operations and Algebraic Thinking

- **Represent and solve problems involving addition and subtraction.**
- **Understand and apply properties of operations and the relationship between addition and subtraction.**
- **Add and subtract within 20.**
- **Work with addition and subtraction equations.**

Number and Operations in Base Ten

- **Extend the counting sequence.**
- **Understand place value.**
- **Use place value understanding and properties of operations to add and subtract.**

Measurement and Data

- **Measure lengths indirectly and by iterating length units.**
- **Tell and write time.**
- **Represent and interpret data.**

Geometry

- **Reason with shapes and their attributes.**

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Operations and Algebraic Thinking 1.OA

Represent and solve problems involving addition and subtraction.

1. Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.[2]
2. Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

Understand and apply properties of operations and the relationship between addition and subtraction.

3. Apply properties of operations as strategies to add and subtract.[3] *Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known. (Commutative property of addition.) To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12. (Associative property of addition.)*
4. Understand subtraction as an unknown-addend problem. *For example, subtract 10 – 8 by finding the number that makes 10 when added to 8.*

Add and subtract within 20.

5. Relate counting to addition and subtraction (e.g., by counting on 2 to add 2).
6. Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 – 4 = 13 – 3 – 1 = 10 – 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 – 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13).

Work with addition and subtraction equations.

7. Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. *For example, which of the following equations are true and which are false? 6 = 6, 7 = 8 – 1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2.*
8. Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations 8 + ? = 11, 5 = ▯ – 3, 6 + 6 = ▯.*

Number and Operations in Base Ten 1.NBT

Extend the counting sequence.

1. Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral.

Understand place value.

2. Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:
   a. 10 can be thought of as a bundle of ten ones — called a “ten.”
   b. The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.
   c. The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).

[2]See Glossary, Table 1.
[3]Students need not use formal terms for these properties.

3. Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols >, =, and <.

Use place value understanding and properties of operations to add and subtract.

4. Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.
5. Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used.
6. Subtract multiples of 10 in the range 10-90 from multiples of 10 in the range 10-90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Measurement and Data 1.MD

Measure lengths indirectly and by iterating length units.

1. Order three objects by length; compare the lengths of two objects indirectly by using a third object.
2. Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. *Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.*

Tell and write time.

3. Tell and write time in hours and half-hours using analog and digital clocks.

Represent and interpret data.

4. Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.

Geometry 1.G

Reason with shapes and their attributes.

1. Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes to possess defining attributes.
2. Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.[4]
3. Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

[4]Students do not need to learn formal names such as "right rectangular prism."

Mathematics | Grade 2

In Grade 2, instructional time should focus on four critical areas: (1) extending understanding of base-ten notation; (2) building fluency with addition and subtraction; (3) using standard units of measure; and (4) describing and analyzing shapes.

(1) Students extend their understanding of the base-ten system. This includes ideas of counting in fives, tens, and multiples of hundreds, tens, and ones, as well as number relationships involving these units, including comparing. Students understand multi-digit numbers (up to 1000) written in base-ten notation, recognizing that the digits in each place represent amounts of thousands, hundreds, tens, or ones (e.g., 853 is 8 hundreds + 5 tens + 3 ones).

(2) Students use their understanding of addition to develop fluency with addition and subtraction within 100. They solve problems within 1000 by applying their understanding of models for addition and subtraction, and they develop, discuss, and use efficient, accurate, and generalizable methods to compute sums and differences of whole numbers in base-ten notation, using their understanding of place value and the properties of operations. They select and accurately apply methods that are appropriate for the context and the numbers involved to mentally calculate sums and differences for numbers with only tens or only hundreds.

(3) Students recognize the need for standard units of measure (centimeter and inch) and they use rulers and other measurement tools with the understanding that linear measure involves an iteration of units. They recognize that the smaller the unit, the more iterations they need to cover a given length.

(4) Students describe and analyze shapes by examining their sides and angles. Students investigate, describe, and reason about decomposing and combining shapes to make other shapes. Through building, drawing, and analyzing two- and three-dimensional shapes, students develop a foundation for understanding area, volume, congruence, similarity, and symmetry in later grades.

Grade 2 Overview

Operations and Algebraic Thinking

- **Represent and solve problems involving addition and subtraction.**
- **Add and subtract within 20.**
- **Work with equal groups of objects to gain foundations for multiplication.**

Number and Operations in Base Ten

- **Understand place value.**
- **Use place value understanding and properties of operations to add and subtract.**

Measurement and Data

- **Measure and estimate lengths in standard units.**
- **Relate addition and subtraction to length.**
- **Work with time and money.**
- **Represent and interpret data.**

Geometry

- **Reason with shapes and their attributes.**

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Operations and Algebraic Thinking 2.OA

Represent and solve problems involving addition and subtraction.

1. Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[1]

Add and subtract within 20.

2. Fluently add and subtract within 20 using mental strategies.[2] By end of Grade 2, know from memory all sums of two one-digit numbers.

Work with equal groups of objects to gain foundations for multiplication.

3. Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.
4. Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.

Number and Operations in Base Ten 2.NBT

Understand place value.

1. Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:
    a. 100 can be thought of as a bundle of ten tens — called a "hundred."
    b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).
2. Count within 1000; skip-count by 5s, 10s, and 100s.
3. Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.
4. Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using >, =, and < symbols to record the results of comparisons.

Use place value understanding and properties of operations to add and subtract.

5. Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
6. Add up to four two-digit numbers using strategies based on place value and properties of operations.
7. Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.
8. Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900.
9. Explain why addition and subtraction strategies work, using place value and the properties of operations.[3]

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[1]See Glossary, Table 1.
[2]See standard 1.OA.6 for a list of mental strategies.
[3]Explanations may be supported by drawings or objects.

Measurement and Data 2.MD

Measure and estimate lengths in standard units.

1. Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.
2. Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.
3. Estimate lengths using units of inches, feet, centimeters, and meters.
4. Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

Relate addition and subtraction to length.

5. Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.
6. Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, ..., and represent whole-number sums and differences within 100 on a number line diagram.

Work with time and money.

7. Tell and write time from analog and digital clocks to the nearest five minutes, using a.m. and p.m.
8. Solve word problems involving dollar bills, quarters, dimes, nickels, and pennies, using $ and ¢ symbols appropriately. *Example: If you have 2 dimes and 3 pennies, how many cents do you have?*

Represent and interpret data.

9. Generate measurement data by measuring lengths of several objects to the nearest whole unit, or by making repeated measurements of the same object. Show the measurements by making a line plot, where the horizontal scale is marked off in whole-number units.
10. Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems[4] using information presented in a bar graph.

Geometry 2.G

Reason with shapes and their attributes.

1. Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.[5] Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.
2. Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.
3. Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

[4]See Glossary, Table 1.
[5]Sizes are compared directly or visually, not compared by measuring.

Mathematics | Grade 3

In Grade 3, instructional time should focus on four critical areas: (1) developing understanding of multiplication and division and strategies for multiplication and division within 100; (2) developing understanding of fractions, especially unit fractions (fractions with numerator 1); (3) developing understanding of the structure of rectangular arrays and of area; and (4) describing and analyzing two-dimensional shapes.

(1) Students develop an understanding of the meanings of multiplication and division of whole numbers through activities and problems involving equal-sized groups, arrays, and area models; multiplication is finding an unknown product, and division is finding an unknown factor in these situations. For equal-sized group situations, division can require finding the unknown number of groups or the unknown group size. Students use properties of operations to calculate products of whole numbers, using increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving single-digit factors. By comparing a variety of solution strategies, students learn the relationship between multiplication and division.

(2) Students develop an understanding of fractions, beginning with unit fractions. Students view fractions in general as being built out of unit fractions, and they use fractions along with visual fraction models to represent parts of a whole. Students understand that the size of a fractional part is relative to the size of the whole. For example, 1/2 of the paint in a small bucket could be less paint than 1/3 of the paint in a larger bucket, but 1/3 of a ribbon is longer than 1/5 of the same ribbon because when the ribbon is divided into 3 equal parts, the parts are longer than when the ribbon is divided into 5 equal parts. Students are able to use fractions to represent numbers equal to, less than, and greater than one. They solve problems that involve comparing fractions by using visual fraction models and strategies based on noticing equal numerators or denominators.

(3) Students recognize area as an attribute of two-dimensional regions. They measure the area of a shape by finding the total number of same-size units of area required to cover the shape without gaps or overlaps, a square with sides of unit length being the standard unit for measuring area. Students understand that rectangular arrays can be decomposed into identical rows or into identical columns. By decomposing rectangles into rectangular arrays of squares, students connect area to multiplication, and justify using multiplication to determine the area of a rectangle.

(4) Students describe, analyze, and compare properties of two-dimensional shapes. They compare and classify shapes by their sides and angles, and connect these with definitions of shapes. Students also relate their fraction work to geometry by expressing the area of part of a shape as a unit fraction of the whole.

Grade 3 Overview

Operations and Algebraic Thinking

- **Represent and solve problems involving multiplication and division.**
- **Understand properties of multiplication and the relationship between multiplication and division.**
- **Multiply and divide within 100.**
- **Solve problems involving the four operations, and identify and explain patterns in arithmetic.**

Number and Operations in Base Ten

- **Use place value understanding and properties of operations to perform multi-digit arithmetic.**

Number and Operations—Fractions

- **Develop understanding of fractions as numbers.**

Measurement and Data

- **Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.**
- **Represent and interpret data.**
- **Geometric measurement: understand concepts of area and relate area to multiplication and to addition.**
- **Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.**

Geometry

- **Reason with shapes and their attributes.**

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Operations and Algebraic Thinking 3.OA

Represent and solve problems involving multiplication and division.

1. Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. *For example, describe a context in which a total number of objects can be expressed as 5 × 7.*
2. Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. *For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8.*
3. Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[1]
4. Determine the unknown whole number in a multiplication or division equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations 8 × ? = 48, 5 = $\square$ ÷ 3, 6 × 6 = ?.*

Understand properties of multiplication and the relationship between multiplication and division.

5. Apply properties of operations as strategies to multiply and divide.[2] *Examples: If 6 × 4 = 24 is known, then 4 × 6 = 24 is also known. (Commutative property of multiplication.) 3 × 5 × 2 can be found by 3 × 5 = 15, then 15 × 2 = 30, or by 5 × 2 = 10, then 3 × 10 = 30. (Associative property of multiplication.) Knowing that 8 × 5 = 40 and 8 × 2 = 16, one can find 8 × 7 as 8 × (5 + 2) = (8 × 5) + (8 × 2) = 40 + 16 = 56. (Distributive property.)*
6. Understand division as an unknown-factor problem. *For example, find 32 ÷ 8 by finding the number that makes 32 when multiplied by 8.*

Multiply and divide within 100.

7. Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

Solve problems involving the four operations, and identify and explain patterns in arithmetic.

8. Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.[3]
9. Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. *For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.*

[1]See Glossary, Table 2.
[2]Students need not use formal terms for these properties.
[3]This standard is limited to problems posed with whole numbers and having whole-number answers; students should know how to perform operations in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

Number and Operations in Base Ten 3.NBT

Use place value understanding and properties of operations to perform multi-digit arithmetic.[4]

1. Use place value understanding to round whole numbers to the nearest 10 or 100.
2. Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.
3. Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations.

Number and Operations—Fractions[5] 3.NF

Develop understanding of fractions as numbers.

1. Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.
2. Understand a fraction as a number on the number line; represent fractions on a number line diagram.
   a. Represent a fraction $1/b$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $1/b$ and that the endpoint of the part based at 0 locates the number $1/b$ on the number line.
   b. Represent a fraction a/b on a number line diagram by marking off a lengths $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.
3. Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.
   a. Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.
   b. Recognize and generate simple equivalent fractions, e.g., 1/2 = 2/4, 4/6 = 2/3. Explain why the fractions are equivalent, e.g., by using a visual fraction model.
   c. Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. *Examples: Express 3 in the form 3 = 3/1; recognize that 6/1 = 6; locate 4/4 and 1 at the same point of a number line diagram.*
   d. Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Measurement and Data 3.MD

Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.

1. Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

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[4]A range of algorithms may be used.
[5]Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.

2. Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l).[6] Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.[7]

Represent and interpret data.

3. Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. *For example, draw a bar graph in which each square in the bar graph might represent 5 pets.*
4. Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.

Geometric measurement: understand concepts of area and relate area to multiplication and to addition.

5. Recognize area as an attribute of plane figures and understand concepts of area measurement.
   a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.
   b. A plane figure which can be covered without gaps or overlaps by *n* unit squares is said to have an area of *n* square units.
6. Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).
7. Relate area to the operations of multiplication and addition.
   a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.
   b. Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.
   c. Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.
   d. Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.

Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.

8. Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

[6]Excludes compound units such as cm^3 and finding the geometric volume of a container.
[7]Excludes multiplicative comparison problems (problems involving notions of "times as much"; see Glossary, Table 2).

Geometry 3.G

Reason with shapes and their attributes.

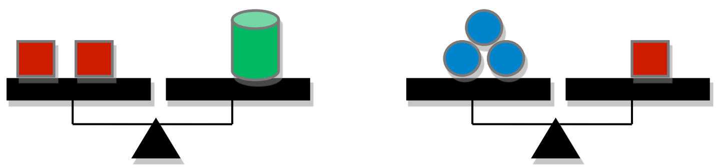
1. Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.

2. Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. *For example, partition a shape into 4 parts with equal area, and describe the area of each part as 1/4 of the area of the shape.*

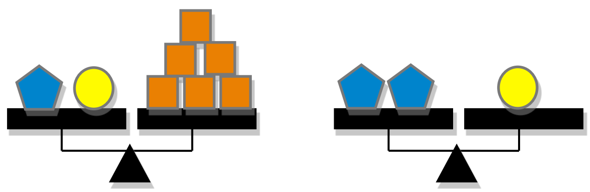
For questions 1 and 2:

a) Which shape weighs the most? The least? Explain how you know.
b) What strategies, reasoning did you use to answer each question?

1)



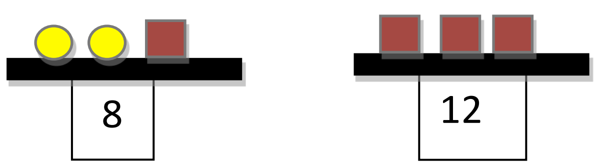
2)



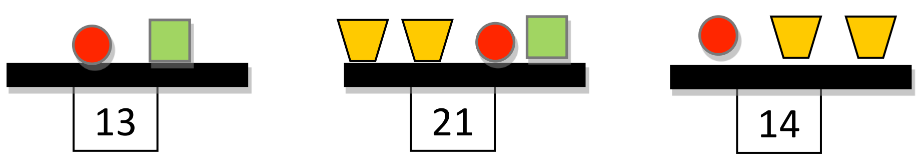
For questions 3 and 4

- Find the value of each shape
- Explain your strategies and reasoning

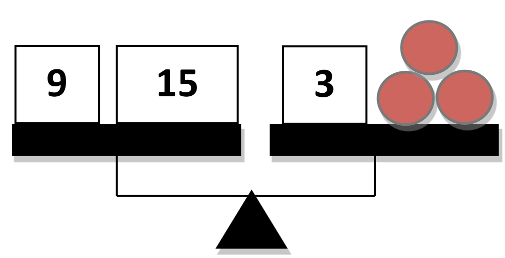
3)



4)



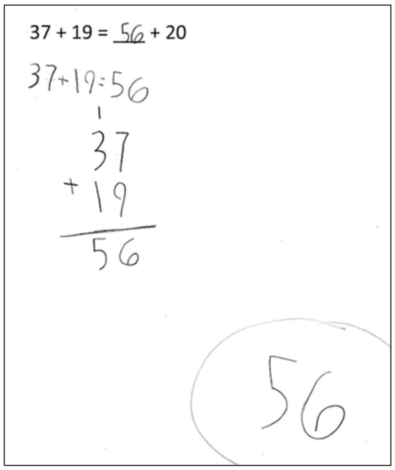
5) Write an equation to represent the balance. Solve.

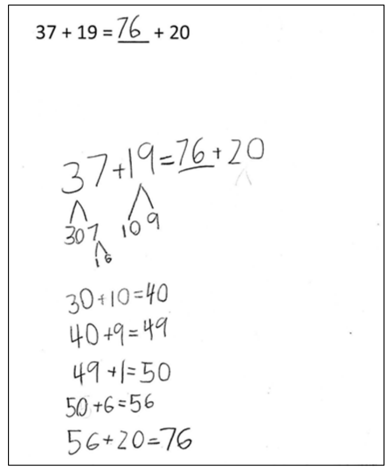


(Kroner, Lou. In the Balance: Algebra Logic Puzzles. McGraw-Hill, 1997)

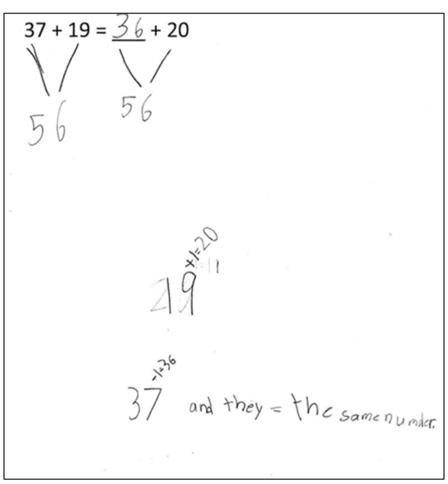
 This product is the result of a collaborative effort between the Ongoing Assessment Project (OGAP) and the Consortium for Policy Research in Education (CPRE) which was funded by the National Science Foundation (DRL-16208888).

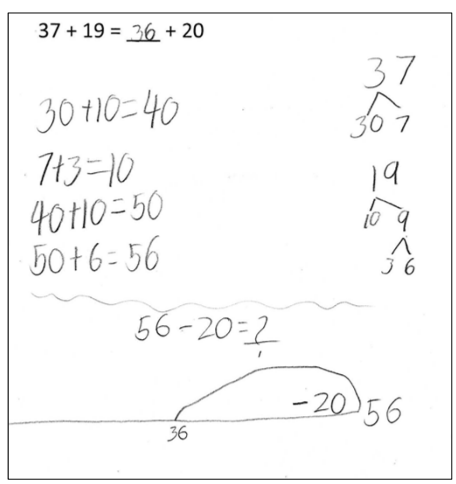
Put a number on the line that makes the equation true. Show or explain how you know.





Put a number on the line that makes the equation true. Show or explain how you know.





Sample Tasks for Equality and Properties

Are these equations true or false? Explain without solving.

Equation	True or False?	Concept or Property highlighted
22 + 18 = 18 + 22		
22 − 18 = 18 − 22		
23 + 7 = 23		
9 + 5 = 14 + 0		
95 + 137 − 137 = 95		
10 – 6 + 6 = 10		
(52 + 4) + 9 = 52 + (4 + 9)		
5 + 7 + 5 = 5 + 5 + 7		
62 + 15 = (62 + 10) + 5		
78 – 49 = 78		
18 – 0 = 18		
0 – 18 = 18		
6 + 9 = 5 + 10		
178 + 24 = 198 + 4		
471 – 382 = 474 – 385		
583 – 529 = 83 – 29		

Open Number Sentences. What number goes in the □ to make the equation true?

Equation	Concept or Property highlighted
96 + 57 = 57 + □	
98 + 74 + 2 = □	
9 + 6 – 6 = □	
75 + 48 + 25 = □	
74 – □= 74	
68 + □ = 57 + 69	

Properties and Relationships: Addition and Subtraction

Properties	
Commutative Property of addition	*a + b = b + a*
Associative Property of addition	*(a + b) + c = a + (b + c)*
Additive identity property of 0	*a + 0 = a*
Relationships	
Compensation	*If a + b = c then (a + x) + (b – x) = c*
Constant Difference	*If a – b = c then (a + x) – (b + x) = c and (a – x) – (b – x) = c*
Inverse relationship between addition and subtraction	*If a + b = c, then c – a = b and c – b = a*

Questions for "Thinking Mathematically" Video 2.3

Equality

1) What is the math goal of the lesson? Does she accomplish the goal?

2) What teaching strategies did she use to encourage thinking and engagement?

3) What was the sequence of the questions she asked? Why do you think she chose this sequence of questions during her lesson?

4) What language does she use to help facilitate understanding of the equal sign?

5) The teacher uses the question (15 + 4 = ___ + 11) to end the lesson. Why?

6) What exit question could you design for her students to check for understanding?

Learning Your Addition Facts

Below is a list of all the addition math facts your child must become automatic with. The facts that are circled are those that your child needs to work on at home and the uncircled facts are those your child already knows. Please help them become more fluent with these facts by taking 10 minutes each day to practice them. We will retest your child in a few weeks and are hoping to see improvement. Knowing these addition facts quickly is essential for success in mathematics this year and in the future.

Thank you for your support.

0 + 0	1 + 1	2 + 2	3 + 3	4 + 4	5 + 5	6 + 6	7 + 7	8 + 8	9 + 9	10+ 10
0 + 1	1+ 2	2+ 3	3+ 4	4+ 5	5+ 6	6+ 7	7+ 8	8+ 9	9+ 10	
0 + 2	1+ 3	2+ 4	3+ 5	4+ 6	5+ 7	6+ 8	7+ 9	8+ 10		
0 + 3	1+ 4	2+ 5	3+ 6	4+ 7	5+ 8	6+ 9	7+ 10			
0 + 4	1+ 5	2+ 6	3+ 7	4+ 8	5+ 9	6+ 10				
0 + 5	1+ 6	2+ 7	3+ 8	4+ 9	5+ 10					
0 + 6	1+ 7	2+ 8	3+ 9	4+ 10						
0 + 7	1+ 8	2+ 9	3+ 10							
0 + 8	1+ 9	2+ 10								
0 + 9	1+ 10									
0 + 10										

Tic-Tac-Toe Sums Game Board

1	24	3	22	5	14
20	8	16	10	12	15
13	23	15	9	17	6
19	7	21	4	23	2
13	20	6	16	17	14
4	19	5	11	18	21

0 1 2 3 4 5 6 7 8 9 10 11 12

Tic-Tac-Toe Sums: Directions

Object: Be the first team to get four sums in a row (horizontally, vertically, or diagonally)

Divide into two teams (X's and O's)

1. Team X selects two addends by placing a marker on the numbers (0-12) to add. The sum is circled by placing an X on the grid.

2. Team O may move one marker to make a new sum and place an O on the grid.

3. Teams alternate moving one marker at a time and continue placing Xs and Os until a team has marked four sums in a row.

4. After several games, players should discuss their strategies.

Digit Delight

Background: This game is designed to engage students in an understanding of place value.

Part I: Playing the Game

Rules:

1) Round 1: The first person rolls the 10-sided die (deltohedron or pentagonal trapezohedron). He/she places a marker on any number on the Digit Delight board that has a digit the same as the number they rolled with the idea being to select a number will the highest value. (e.g., a 9 is rolled. I can choose between 985 and 609. I choose 985 since the 9 is worth 900 in 985 and only 9 in 609). Once the choice is made the player records the turn on the Digit Delight record sheet. The second player does the same thing on their Digit Delight board.
2) Subsequent rounds: Again a player rolls the die. He/she can move their marker only to an adjacent space with the digit they rolled. (e.g., my marker is located as below (star). I roll a "6." There are 3 choices of adjacent numbers (including the diagonal) that have a digit with a 6 in it – 867, 960, and 613. I choose 613 because the 6 has the greatest value. The player records 600 in the record sheet and so on.

3) The winner of the game is the person with the most points after 10 turns..

Digit Delight Record Sheet

Turn	Digit Rolled	Number Selected	Value of Digit	Running Total
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

Digit Delight Board

130	713	542	985	425	249	476
316	308	867	960	547	609	176
713	542	958	542	590	476	248
790	613	591	349	830	671	258
824	179	645	825	325	806	718
956	417	832	470	879	102	910
130	238	382	630	407	315	340

Sum What

Goal: To get the lowest score.

Directions:
Roll 2 dice.
Add the 2 numbers to get a total.
Cover any number or combination of numbers that equals that total.
Keep doing this until the first time you roll and can't cover anything.
Find the total of the uncovered numbers.
The other player can continue until they can't cover any numbers.
The player with the lowest score wins.

1	2	3	4	5	6	7	8	9

1	2	3	4	5	6	7	8	9